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Parents Nurturing Math-Talented Young Children

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Nancy M. Robinson
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University of Washington
Seattle, Washington

December 1996
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By far, our biggest debt goes to the participating children and their parents. We learned much from observing and teaching these lively, spirited children. They taught us, too.

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ABSTRACT

Talent in mathematical reasoning is highly valued in this society, and yet very little is known about its early course. This book is an outgrowth of a two-year study of children discovered during preschool or kindergarten to be advanced in their thinking about math. Among other findings, the study revealed that, as a group, the children remained advanced in math over the two-year period, that their spatial reasoning related more closely to their math reasoning than did their verbal reasoning (although they were ahead in all three domains), and that the math scores of the boys started and remained somewhat higher than those of the girls. The biweekly Saturday Clubs to which half the group were randomly assigned also proved effective in enhancing mathematical reasoning.

The children were identified originally by their parents, and although it was clear that they must already have been nurturing their children's development in effective ways, the parents had many questions about how they might further enhance mathematical development at home. The ideas presented in this book grew out of our Saturday Club experiences as they might translate to family activities. The book describes characteristics of math-advanced young children; ways to "tune into" children's ideas and questions through informal play without becoming didactic or turning off their curiosity by drilling number facts and procedures; and the power of "big ideas" like infinity, zero, reversibility, equivalence, representing the numeration system in different ways, measuring, estimating, gathering data, and understanding probability. A wide assortment of real-life contexts, such as gardening, cooking, planning parties, dealing with money, going out to eat, caring for pets, making collections, and car tips, are also described as the occasions for mathematical explorations. A final chapter presents a variety of alternatives by which schools and parents, working in partnership, can create optimal ways to support the development of highly capable children.

Table of Contents

Table of Contents (continued)

Table of Contents (continued)

List of Figures

Parents Nurturing Math-Talented Young Children

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CHAPTER 1: What Do Math-Talented Young Children Look Like? How Is a Parent to Know?

Introduction

The development of children's mathematical thinking is a wonderful thing to watch. Almost before their very eyes parents can watch mathematical ideas and counting skills become part of a child's vocabulary and repertoire. These mathematical ideas help to organize a child's mental world, make sense of experience, and quantify the important data in a child's life. For many children, mathematical ideas and their potential to organize and deepen experiences and activities become a source of profound pleasure and wonderment. These children express an affinity for the number system and how shapes work together to form other shapes, delight in playing with numbers and finding patterns, and ask big questions about the more profound aspects of mathematics.

This book is about ways in which parents can support (and enjoy) the development of young children, roughly ages 4 to 7 years, who are good at math. It is an outgrowth of a two-year study of children we discovered as preschoolers or kindergartners to be advanced in their thinking about math. With the support of a Javits grant from the Office of Educational Research and Improvement of the U.S. Department of Education, we studied 300 children over a period of two years and involved half of them in biweekly Saturday Clubs designed to enrich their experience with mathematics. The study, which we named "Math Trek," will be described in greater detail in the next chapter. It furnished us with the basis for this book because, in the course of this investigation, these varied and lively children and their families taught us so much we hadn't known. Indeed, it was the first study to look in any systematic way at young, math-talented children, so there were a lot of questions to be answered!

So much of children's development is informal, on-the-fly, engaged in real-world discoveries, that parents are, indeed, participants in the most important learning of all. Such interactions can be very special for all concerned. Perhaps more than anything, this book aims to reassure parents about the roles they can play in nurturing their children's mathematical thinking and to give them permission to enjoy themselves in the process.

Math-Talented Children Are a Varied Group

Young children who are good at math range from those who are "pretty good" at math to those who are so remarkably advanced that adults find their ideas quite astonishing. The children differ in many other ways as well. Some are better at thinking with words while others work their ideas out better in a visual-spatial manner, with objects and drawings. In some children, advancement in mathematical reasoning is just one part of a picture of high ability in many domains, and in some others, math reasoning seems more specialized, more advanced than other skills.

Seth was clearly a "numbers boy." The first Saturday Club, he immediately appropriated a set of dice with numbers on some sides and plus and minus signs on others and made a trail the length of quite a long table, alternating numbers and signs. With considerably greater speed than the adult who was trying to keep up with him, he calculated the entire series all the way to the end, including "magic" numbers (negative numbers that could, he explained, "zap" [cancel] other numbers). Silent and prone to work alone, Seth was engaged with challenging math and science activities that "looked like" math but lost interest when we read or acted out stories that embedded mathematical concepts.

In contrast, Laura could be counted on to engage in any activities that were sufficiently challenging, no matter what their form. She tended to pick the hardest and most novel of the tasks suggested on the job cards the children found waiting for them on arrival, and her journals were full of carefully labeled drawings and diagrams. "What's a mathematician?" we asked one day. "A thinker," said Laura. "That's me!"

Some of the children, we found, are deeply passionate about numbers. They ply their parents with questions about the numbers they see around the house, on the road, at the fast-food restaurant, on the television set. They may carry around a problem in their heads, mulling it over at odd times until it makes sense to them. They make up challenging problems for themselves and prefer hard problems they can't answer right away to easy ones for which they have a ready answer. Their smiles of satisfaction when they have figured something out are a pleasure to see.

When we met Johnny, he was confident and nodded enthusiastically when we said to him, "We hear that you're a boy who likes math." During the testing session in which we screened nominees to determine the final participants in Math Trek, it was when we got to the harder questions that his eyes really began to shine. Although he didn't know many number facts, he was persistent in working out complex problems using his fingers, sometimes drawing imaginary pictures in the air to help keep track. One could see him "inventing" new strategies as he went along, like decomposing numbers into their parts so that he could add them more easily. The stickers we gave Johnny for completing subtests were fine with him, but it was clear that the real fun was in the figuring.

Other children, even quite young ones, despite their advanced abilities in math, don't get the same fun out of numbers. They may have been drilled too early in math facts; they may have been exposed to siblings, teachers, or parents who see math as "boring" and who don't have a love of numbers themselves; or they may feel some pressure—from others or from themselves—to get things right the first time, with no mistakes along the way. These children don't spend much time asking math-related questions or making up problems for themselves. Lacking confidence, they tend to prefer easy problems to hard ones and in school, they tend to shy away from activities that are labeled "math."

Sam was solemn and wary when we first met him for the screening session. His father had told us that Sam was doing some very advanced calculations at home under his tutelage, and seemed quite proud that Sam was now learning his "17-times tables." To the comment, "We hear that you're a boy who likes math," however, Sam barely nodded. Even after we got to work, he seldom smiled, although he commented with some relief at the early problems, "These are easy!" As soon we proceeded to harder ones, he was ready to quit. Sam used none of the finger-counting strategies young children usually do; when faced with a problem that he labeled "multiplication," but to which he could not immediately call up the answer, he did not try adding the numbers, even when encouraged to "figure it out." Story problems brought major confusion; to him, math implied paper-and-pencil calculation. It was clear that, despite his advanced skills, Sam did not see himself as a "mathematician." Anything but.

Characteristics of Math-Talented Children

One of the things we did when parents nominated their children for Math Trek was to ask them a general question as to what clued them in to the observation that their child was advanced in math. When we looked at answers given by parents of some children who subsequently scored quite high on our screening measures, here is what the parents said:

At four years old, he could identify all the states of the U.S. by shape alone and place them appropriately in a puzzle without outline cues.

Likes to organize collections of things and memorizes details, including numerical information, e.g., baseball cards . . . license plates.

We haven't paid much attention to his mathematical ability . . . mostly because his language skills are extraordinary. He reads, at least, at a fifth grade level.

At about 4 1/2, (son) began to play with numbers (counting by 2s, 3s, 4s, etc.), timed his ability to count (31 minutes to reach 1000), distinguished between even and uneven numbers ("Some numbers don't have middles.")

Yesterday, at home, he discovered the internal shapes within various stars (a square within a four-pointed star, a pentagon within a five-pointed star, etc.). (Son) absorbs anything he can learn about math because, to him, it is fun.

When I mentioned the possibility of participating in this project, she was quite excited, remarking, "Good. I get to do some stuff that isn't plain."

At 3 1/2 he could estimate another person's line of sight.

Using a stopwatch to time parts of a trip.

Has recently shown interest in written music—how notes and rests divide a measure.

Continues to ask many conceptual math questions about things like: positive/negative numbers, infinity, fractions, etc.

Complains that they don't do math in kindergarten.

(Daughter) is in love with math. She walks around looking for "square numbers and paladromes (sic)."

Discovered the Montessori math teacher's manual last spring.

She and her father had a lengthy discussion on Avogadro's number, which is now called Avocado's number. She can tell time and can write Roman numerals up to 20 easily.

Will multiply and divide using factors up to 10 and various combinations of numbers. All this is done in his head . . . the process is what interests him. (Pletan, Robinson, Berninger, & Abbott, 1995, pp. 35-36).

Just like any other group of children, math-talented children are a very diverse group. Not only do they differ in obvious ways, such as gender and ethnic background, but they differ in the ways they approach mathematical thinking. The following list represents characteristics observed by elementary school teachers who have participated in Math Trek workshops and descriptions listed by House (1987, pp. 51-52). Here are some aspects of math-talented children's development a parent might notice (though no single child is likely to exhibit all of them):

- Awareness of numbers in their surroundings and a tendency to frame questions numerically ("How many minutes to lunch?" "I want 7 grapes." "What day is today?" "What time is it?"); children often count spontaneously, such as stairsteps as they go up or down, or the blocks they have assembled in a Duplo tower
- Early-appearing ability to play games such as Parcheesi, Trouble, chess, or even Monopoly that involve strategic play (e.g., which piece to move or which hotel to buy; anticipating when one's own or one's opponent's piece is endangered)
- Interest in money—making small purchases or using a pay phone independently, making change, keeping track of an allowance, asking prices of desired items
- Interest in numbers on the highway—tracking the speed limit signs and the speedometer ("Mom, you're driving too fast!), asking about highway numbers or freeway exit numbers, reading "miles to" signs
- Early interest in learning to tell time
- Enjoyment of challenging mathematical puzzles and games or computer programs
- Advanced knowledge of number facts and above-grade ability to do written computations (not necessarily advanced problem-solving skills)
- Advanced problem-solving skills (not necessarily advanced knowledge of number facts or of written computations)
- Exceptional mathematical memory—ability to keep in mind the multiple steps of a problem
- Ability to hold problems in mind that aren't yet figured out, to ponder them from time to time until the answer emerges
- Wanting to know what mathematical symbols mean and wanting to write them down
- Asking for dot-to-dot and/or math workbooks to do for fun
- Making collections (baseball cards, stamps, rocks); counting and categorizing
- "Number sense"—a ballpark "feel" for whether an answer is reasonable, for whether a procedure might be appropriate
- Frequent step-skipping in problem solving, and unexpected ways of solving problems; capacity for inventing strategies that need not be the conventional ones
- Rapid and intuitive understanding—thinking faster than they can write down their answers or describe how they got there

- When presented with a choice, choosing activities related to mathematics (around home, for example, measuring the ingredients for a cake; playing "store" with prices; playing "school" with numbers rather than letters; planning an elaborate train track; reading the thermometer before planning what to wear, etc.)
- Keeping track of one's own life and activities (e.g., how much of weekly allowance has been saved; how many days until birthday)
- Interest in looking for patterns and relationships and explaining them (for example, awareness of the shapes in a quilt or a pattern in the pavement; comparing stars with different numbers of points)
- Fluency in representing mathematical ideas in different ways—e.g., Legos or pattern blocks, drawings, equations
- Long periods of absorption with problems in which they are truly engaged; reluctance to give up on an unsolved problem
- Treating conceptual road-blocks as challenges; detouring rather than retreating in the face of obstacles; "courageousness" in trying new pathways of mathematical thinking
- Eagerness to find connections between a new problem and problems previously solved or ideas from an entirely different domain
- Pleasure in posing original, difficult problems
- Joy in working with "big" numbers
- Capacity for independent, self-directed activities that have to do with math

Of course, as we've said, no single mathematically talented child is likely to show all of these characteristics. Nor is the list exhaustive.

Children also differ in their degree of advancement, so that some children's abilities are quite astonishing, while others are more subtle. But if parents suspect that their child does have a special interest or talent in mathematics, it might be worth while keeping an informal diary for a little while in order to become more aware of the child's personal approach to this important domain.

Is there any reason to rush out and have your child evaluated to see whether in fact your suspicions are correct that his or her development is advanced in this domain? Not at all! Not unless you have a sense that your child's school situation is less than a good fit for his or her needs in this respect (see Chapter 6). For other purposes, you needn't "prove" anything about your child's development—you need only observe and respond accordingly.

As we will discuss in the next chapter, parents are usually right when they suspect that their children are ahead in one or more domains of development. Of course, parents who are not familiar with a lot of other children may not recognize that their child's mathematical questions and passions are different from those of other children, but, even so, effective parents are sensitive to their children's development at any given moment—and that's quite enough.

The rest of this book is directed mostly toward ways in which parents can support their children's mathematical endeavors at home. Whether children are one year or four years ahead of agemates in mathematical thinking will be much less important at home than whether they are encountering new experiences that let them extend their ideas about the number system, that require a bit of time to figure out, and that make them want to know more rather than less.

Many parents don't come to the role of math-guide effortlessly. In some homes, to be sure, numbers are so much a part of everyone's thinking that much of what we will describe in this book comes very naturally. In those homes, children are introduced very early to notions of counting, size, and shape in playful ways as they come up in everyday life. In other homes, families regard mathematics as very important and are planful about their teaching. Sometimes, as a consequence, they introduce their children too early and too vigorously to written calculations, number facts, and right/wrong answers without the underlying ideas, the experimentation, or the playfulness that enable young children to "own" what they are learning. Some parents, even those whose occupations are very number-oriented, don't share number play with their children. Still other parents, whose own school encounters with math were not particularly pleasant, think their children are "too young to have to think about math." Some parents simply talk a lot more to their children than others do, even though they may be together an equal amount of time. And some families live such stressed-out, chaotic lives that children don't experience the daily routines and predictability that would lead them to believe that a system of any kind, such as the number system, really exists.

Since the late 1980s, significant reforms have taken place in teaching mathematics to children at all levels. These reforms, led by the National Council of Teachers of Mathematics (1989), have moved mathematics instruction into the real world and emphasize children's number sense rather than rote practice with computational operations and memorizing number facts. They encourage children's ability to grasp meaning in the numeric system, to see patterns and relationships, to solve real problems using appropriate strategies, and to communicate their findings and discoveries to others. Instructional approaches these days involve small groups of children problem-solving together, using a variety of materials (different sorts of blocks and shapes, formal and informal devices for measuring, calculators, computers); talking, writing, and drawing about their ideas; examining the ways in which their ideas differ from those of their classmates; developing their own questions and problems; and otherwise engaging in activities that seem very different from the work-sheet approaches of yesterday that were all, not just a part, of math instruction.

Such an instructional approach enables teachers to see what children are thinking, to become question-askers rather than answer-tellers, and to help children become engaged in problems that are meaningful to them. This is an approach that most parents will find compatible with good parenting: more interactive and less "teacherish," allowing adults to become intimate with the way children are thinking at any given phase of their development, and engaging children in the everyday kinds of activities that life is made up of. It is the kind of atmosphere in which children's thinking thrives, at home or at school.

Chapter 2 will describe our Math Trek study in some detail. In Chapters 3 through 5, we will describe some ways in which parents can promote the conceptual development and learning of their math-talented young children through open-ended questions and activities that are a part of everyday life. Finally, in Chapter 6, we will discuss ways in which parents can intersect with their children's school experience in order to achieve the best possible match between their children's developmental levels and their school experience.

These are wonderful children—and they didn't get that way by themselves. Both nature and nurture have played significant roles in shaping their abilities and interests. Parents can rest assured that they must have been doing a great deal that is just right if their children have come to the ripe old age of 4, 5, 6, or 7, already advanced in mathematical thinking!

CHAPTER 2: Math Trek—Identifying and Nurturing Mathematical Precocity in Young Children

The early discovery of talent is the first step toward helping it grow. There are surely many children with special talents whose abilities are doomed to wither for lack of attention and encouragement (Feldman, 1986). Indeed, in the case of mathematical talent, advanced ability can be actively discouraged if children are forced to repeat, over and over, low-level skills they have mastered long before, as too often happens in school classes in which everyone does the same thing at the same time. Children's love of math can also wither if they are taught that mathematical thinking consists of memorizing number facts or doing written calculations without the playfulness that is their natural inclination in dealing with ideas they love. As we have just seen, math-advanced children show their abilities in a variety of ways. It is up to the adults who nurture them—teachers and parents—to identify and engage children in expanding their interests and competencies if their advancement is not to be lost.

Talent in mathematical reasoning is highly valued in this society and is basic to many career paths, especially those leading to science and technology. Yet, little is known about the very early course of mathematical talent, and virtually no energy has been expended in searching for young children who are good at math. For older students who reason well mathematically, annual regional talent searches (Stanley, 1990) now involve some 160,000 or more seventh-graders who test their skills with high-level academic aptitude measures, namely, the *Scholastic Aptitude Test* (*SAT*) and the *American College Test* (*ACT*). In some regions, there are talent searches for upper elementary school students using eighth-grade level tests, and there are math contests of various kinds for junior and senior high school students. But math-talented children in the primary grades remain a mystery. Math Trek was directed toward finding out more about them.

Research About Math-Talented Students

Most of the research on math-talented students and their thinking has also focused on the teenage years. The research has been possible in large part because of the access investigators have had to the participants in the talent searches mentioned above. A few investigators (Assouline & Lupkowski, 1992; Lupkowski-Shoplik, Sayler, & Assouline, 1993; Mills, Ablard, & Stumpf, 1993) have looked at older elementary students who are good at math, but no one has really looked at math-advanced children as young as the early primary grades.

Yet, learning about numbers begins well before children enter school. It is a process that emerges both in number-related social activities with parents and peers (Saxe, Guberman, & Gearhart, 1987) and in the children's self-directed play. Many children play with objects that are structured to have regular relationships to one another, such as blocks of different sizes, train track pieces, Legos or Duplos, or simply groups of small objects such as cars, bugs, or crayons (Ginsburg, 1989). Some of the children's first language achievements have to do with quantities; "more," "all gone," and "big girl" are some of the first words they learn. "Are we there yet?" leads to all kinds of conversations about time and distance. Cooking, going to the store, reading books, setting the table, even finding the right channel on the TV are all number activities that come naturally at home. Preschoolers—even babies—know a great deal about numbers even if they do not fully grasp the concepts (Gelman & Gallistel, 1978).

About the time children enter school, the 5- to 7-year shift occurs, a benchmark era during which children's thinking normally becomes systematic and stable enough to be consistent from one specific context to another (Sameroff & McDonough, 1994). Very young children may have good mastery of an idea in one context but not in another. Children who have figured out that two books and three books are five books may not understand that three forks and two forks are five. Between five and seven years, children also begin to take off in their mathematical reasoning and to grasp the notion that numbers have reliable, systematic relationships to one another and mean the same thing wherever you find them.

When children first begin to use numbers, they have to figure things out every time. Over and over, they may add four and three on their fingers to get seven. Most children continue to engage in slow, effortful computation during the primary grades. Not until fourth grade do children usually begin to show habitual automaticity, that is, apparent effortless recall, in retrieving basic number facts (Kaye, de Winstanley, Chen, & Bonnefil, 1989).

Despite this extensive interest in the normal course of children's understanding of numbers, hardly any investigators have looked at differences in the rates with which they acquire their reasoning skills and number knowledge. Certainly this is true of the early ages in which we are interested. Although most research on the development of early mathematical thinking has ignored differences among children, some research has looked at distinctive characteristics in strategy use (e.g., Geary & Brown, 1991; Siegler, 1995).

Even by first grade, one can see some individual differences in the strategies children use (Siegler, 1991, 1995). Nearly all beginners use a variety of strategies that are relatively effortful and often inaccurate (e.g., "counting all" in adding 2 + 3 (1, 2, 3, 4, 5) versus "counting on" (2 + 3, 4, 5) versus a "min strategy" that adds the smaller to the larger (3 + 4, 5). With experience, however, they shift to more mature retrieval strategies, calling upon facts they know rather than figuring them out each time (Siegler, 1991). Gifted children, being more like older children in their problem-solving, use retrieval more frequently than do average children of their age (Geary & Brown, 1991).

Even among children who are good at math, personality differences can play a role in the way they deal with solving problems. For example, Siegler (1988) described three types of first-graders: "good" students, "not-so-good" students, and "perfectionists." The perfectionists, who wanted to make sure they were right, performed as well as did the good students but had higher standards and did much more checking than the other two groups, even though they could retrieve number facts from memory as well as the good students.

Research Questions

The study from which this volume grew had a number of facets. So little was known about young children who are good at math that there were many questions to answer. Among them:

1. Can we locate young children who are good at math? Can parents pick them out? Will parents and test scores agree?
2. If young children are good at math, what else are they good at? What kinds of cognitive abilities go along, at these early ages, with the ability to reason well in the quantitative domain?
3. Are there gender differences in math precocity at this young age, or do they not appear until later? What does entering school do to any gender differences that are observed?
4. How stable is mathematical precocity? Will young children who are initially good at math retain their rapid pace of development in mathematical reasoning?
5. And finally, if children are given extra experience with mathematical thinking in a friendly and engaging environment that invites inquiry, will the intervention affect their mathematical reasoning? Their attitudes toward math?

The Research Team

And so Math Trek was born, with the help of funding from a Javits grant for research on the gifted and talented, Office of Educational Research and Improvement, U. S. Department of Education. Math Trek was a complex effort by a team that included several senior investigators but also a multitude of teachers and student assistants who contributed significantly to its outcomes.

The several senior investigators, most but not all from the University of Washington (UW), came to the project with different backgrounds and agendas. Nancy Robinson, Ph.D., UW Professor of Psychiatry and Behavioral Sciences and Director of the Halbert Robinson Center for the Study of Capable Youth, brought a long interest in the early emergence of precocity in development and a knowledge of psychological testing. For example, with several colleagues, she had previously followed to school age early-talking toddlers (Crain-Thoreson & Dale, 1992; Robinson, Dale, & Landesman, 1990) and other preschoolers nominated by their parents as precocious in any of a variety of domains (Robinson & Robinson, 1992), using ability tests to document their progress. Virginia Berninger, Ph.D., Professor of Educational Psychology and head of the UW School Psychology Program, brought further expertise in psychological measurement, individual differences, and research methodology. Robert Abbott, Ph.D., UW Professor of Educational Psychology, brought a strong background in conceptualizing and analyzing complex sets of psychometric data. Yukari Okamoto, Ph.D., Assistant Professor of Psychology at the University of California at Santa Barbara, brought a background of neo-Piagetian theory and measurement of children's conceptual structures, as did Robbie Case, Ph.D., of the Ontario (Canada) Institute for Studies in Education. These investigators constituted the part of the team that directed major efforts at measuring and analyzing the cognitive aspects of the children's development.

Other members of the team brought expertise in teaching and focused on the intervention aspects of the project, the Saturday Clubs. Swapna Mukhopadhyay, Ph.D., UW Assistant Professor of Curriculum and Instruction, is a specialist in teaching young children mathematics, and provided much of the inspiration for the approach we used and much of what will appear in succeeding pages. Barbara Waxman, Ph.D., during these years was completing her dissertation on young children's reasoning about math, work she had begun years earlier at the University of Rochester. Barbara was head teacher in the Saturday Clubs and, as authorship of the volume attests, became its voice as well.

The Research Plan

In the spring of 1993, a publicity campaign was mounted throughout the Puget Sound region, the metropolitan area surrounding Seattle, to find young children, then of preschool and kindergarten age, who were thought by their parents and/or teachers to be "good at math." Letters to preschools and schools, meetings with Head Start teachers and those involved in a similar state-funded program, articles in local newspapers, and radio talk-show interviews were the means we used. Rough guidelines were mentioned such as, "Asks questions about numbers or time." "Makes up games using numbers, such as playing store with prices." For preschoolers: "Uses adding and subtracting up to 5 and understands that these are related; knows that a dime is more than a nickel; plays board games involving counting spaces." For kindergartners: "Makes small purchases; wants to learn to tell time; reads symbols such as plus and minus; reads speed-limit signs; may understand that multiplication is shorthand for adding."

The identification process resulted in the nomination of 798 children, with 778 families actually mailing in information and consent forms and permitting their children to be screened. In the screening process, testers administered the Arithmetic subtest of the *Kaufman Assessment Battery for Children* (*K-ABC*) and the Arithmetic subtest of the *Wechsler Preschool and Primary Scale of Intelligence, Revised* (*WPPSI-R*). Children who had reached their sixth birthdays and who hadn't topped out on the *WPPSI-R*, were also given the Arithmetic subtest of the *Wechsler Intelligence Scale for Children*, *Third Edition* (*WISC-III*). All of these subtests asked the children to do mental math to solve brief story problems.

From the total group of children we screened, we first identified all the children who had attained a score at the 98th percentile or higher on any of the screening measures. We also included four boys with lower scores but compelling evidence of special interest in numbers. Of the 348 children who met these inclusion criteria, we kept all the girls but randomly dropped 9 preschool and 29 kindergarten boys. The final sample of 310 children included 61 preschool girls, 78 preschool boys, 77 kindergarten girls, and 94 kindergarten boys. That fall, they attended a total of 157 different elementary schools, and two children were home schooled. The children were then randomly assigned to either a comparison or an intervention group, with the provision that all the children who attended a single school were in the same group.

One arm of the study was directed at following the cognitive development of all 310 children for two years. In the fall of 1993, and again in the spring of 1995, the children were individually administered a battery of measures tapping not only a wide variety of mathematical reasoning functions but also verbal ability, visual-spatial ability, and short-term working memory span in the verbal, visual-spatial, and mathematical domains. Each time, the children were asked questions about their views of their academic competence that were taken from the *Pictorial Scale of Self Perception* (Harter & Pike, 1984). In the final round, we also asked questions about what they found satisfying in math, some of these our own queries and some borrowed from Nicholls, Cobb, Wood, Yackel, and Patashnick (1990). Embedded in the battery were experimental measures of the children's maturity in conceptual thinking that had been developed by neo-Piagetian investigators (Case, 1985; Crammond, 1992; Okamoto, 1992a, 1992b; Okamoto & Case, 1996); these measures as well as the Harter questions were also administered to the children half-way through the study, so that special analyses of their growth patterns could occur.

The other arm of the study is of greatest relevance to this book. For the children in the intervention group, Saturday Clubs were offered every other Saturday during the two succeeding school years, a total of 28 sessions in all. In groups of about 15, with the guidance of certified elementary teachers most of whom had been trained by Professor Mukhopadhyay, children met for half-day sessions, either morning or afternoon. In these sessions, the children were offered opportunities to engage in a wide variety of math activities. Most teachers were assisted by two or more assistants, some of whom were graduate-student teachers in training and some of whom were undergraduate psychology students. Various sites were provided to be as geographically convenient as possible for this far-flung group of families. For the most part, the younger group (who were in kindergarten and then first-grade as the study progressed) met in the morning and the older group (first-grade and then second-grade as time went on) met in the afternoon, but adjustments were made for the children's other activities (soccer games, ballet lessons, and chess tournaments were our major competition), car pools, and, in some cases, maturity levels. There will be much more about the Saturday Clubs in the chapters to come.

Although Saturday Clubs were the most important part of the intervention, we also talked with the teachers of most of the children in the intervention group at least once during the year, both to understand more about how the children were doing in school and to brainstorm with teachers some additional adaptations of classroom procedures that might be appropriately challenging for these math-advanced children. Some teachers were already beautifully in tune with the needs of the children and successfully compacting the curriculum and making a variety of more challenging opportunities available. (See Chapter 6.) We also held parent meetings each year in which we shared what we were finding from the Math Trek project, engaged in some hands-on tasks, and displayed books and materials that parents might find useful at home.

Research Findings

Could we readily locate math-talented children? We had conducted other studies in our laboratory in which we had recruited young children by community appeals. We knew from these previous studies that parents of very young children, even toddlers, could identify children who were advanced in talking, reading, and general intelligence, but we weren't sure that, before children encountered formal math in first grade, their parents would tend to notice their math talents. The results of our initial search showed that, indeed, parents can identify math-talented children. On average, for the 778 children we screened, mean standard scores on the Arithmetic subtests of the *K-ABC* and the *WPPSI-R* placed them at the 86th to the 90th percentile. Had parents not been good at identifying the children, the mean standard score should have been at the 50th percentile. The children we picked to follow as a group of course scored even higher. Their mean standard scores on the *K-ABC* arithmetic subtest was at approximately the 95th percentile and their mean standard score on the *WPPSI-R* arithmetic subtest placed them at about the 98th percentile. In the total nominated group, there were plenty of such children (except for preschool girls, as we shall see). About half of the children who were nominated, indeed, met this more stringent criterion.

To investigate parents' ability to identify math-talented children, a sub-study was conducted by one of our students (Pletan et al., 1995). Pletan sent a questionnaire to the first 120 families who had nominated kindergarten children. He developed the questionnaire on the basis of the comments about their children's early interest in mathematical ideas that parents had been asked to include when nominating the children.

Pletan's study demonstrated that, even prior to their children's entering first grade, parents were good at picking out math-advanced children and at describing their skills. Five clusters of abilities seemed to clue the parents as to their children's advancement: a general intellectual factor, a short- and long-term memory factor, a rote (rehearsed) memory factor, a spatial reasoning factor, and a factor reflecting children's specific knowledge of number relations. Of course, not all children were equally advanced in all these clusters of abilities. We found, moreover, that the parents' overall observations of their children's thinking about mathematical issues showed a strong relationship to the children's performance on the screening measures we used. Indeed, the parents' descriptions could pick out those who were dramatically advanced versus those who were more mildly advanced. This finding strengthened our assumption that the tests were a valid short-cut to finding children whose real-life abilities and interests in mathematics were ahead of their peers'. The screening measures that were used were carefully chosen to reflect the real-life mathematical problems that young children are easily engaged in. For instance, one of them utilized the context of a visit to a zoo, posing increasingly complex questions concerning buying peanuts and getting change.

At what other cognitive tasks are young, math-talented children advanced? We were not surprised to find that the children were advanced on all the standardized measures we administered, not just the math subtests. Their scores on the measures of verbal ability, the Vocabulary, Comprehension, and Memory for Sentences subtests of the *Stanford-Binet, Fourth Edition*, hovered around the 90th percentile. The same was true of their scores on the two *Stanford-Binet* subtests that tap visual-spatial reasoning, Pattern Analysis and Matrices.

Within this pattern of advanced abilities, however, some abilities were more closely related to math than others. Specifically, the children's scores on the various visual-spatial measures (the two above, plus a visual-spatial working-memory span measure) were much more closely correlated with their math reasoning scores than were their scores on the verbal measures. This was a little less true for the older group than the younger group, but the pattern was the same. For young children who are good at math, visual-spatial reasoning abilities probably play a very important role in the way they think about math, a more important role than does reasoning in words or purely verbal knowledge.

Are there gender differences even at this early age? Alas, there are. More boys than girls were nominated for the study, and of the children nominated, the boys tended to score higher. Indeed, within our group we had a hard time finding as many girls in the younger group as we wanted despite extra appeals. As mentioned previously, we finally settled for an incomplete group (61 versus the 77 we had hoped for). In contrast, we found a superfluity of boys for the older group. And, within the battery of scales we administered, there were significant gender differences in favor of the boys on seven of nine math reasoning measures, including one memory measure; not on the visual-spatial reasoning measures but on the one visual-spatial memory measure; and not on either the verbal reasoning or the verbal memory measures. When we looked at just the highest-scoring children, those in roughly the top 5% on each measure, the same patterns held—the boys were again overrepresented in this top group on the mathematical reasoning measures. The pictures for the verbal and visual-spatial measures were more mixed and not gender-lopsided.

We didn't know what to make of this finding. We certainly had no definitive explanation of the gender differences. We speculated that, whether or not there were any built-in gender differences in mathematical reasoning, the children's life experiences might well have influenced their number knowledge and number sense. Little boys, it's easy to observe, tend to gravitate in their play to countable and structured materials like blocks, cars, and railroad tracks; little girls tend to be more social in their play (Maccoby & Jacklin, 1974). We suspected that parents might tend to engage sons more frequently than daughters in talk about numbers. Is it possible that the gender differences we saw were wholly the product of children's differential experience? Could some built-in differences have been magnified during the preschool era?

How stable were the findings of the first year? We waited anxiously to see how the children fared over the next two years, when they were tested again at the end of first and second grade. Indeed, in all three domains we examined (verbal, visual-spatial, and mathematical), and including the comparison children not in the Saturday Clubs, not only remained as advanced over their agemates as they had been the first year, on some mathematical reasoning measures, they were even more decidedly ahead. Among the standardized subtests on which we could compare them with children of their age in the general population, at the last testing they were even more impressive in their abilities to extend series of numbers arranged according to principles they had to discern, to visualize mathematical problems spatially, to answer questions about the number system, and to do written calculations. In visual-spatial reasoning tasks such as copying block patterns or completing matrices, they had also made advances relative to their age peers. We concluded that not only were these children ahead of their peers when we first saw them, as a group, they held their own in every respect and made even more impressive gains over the next couple of years.

What about the gender differences? Alas, these did not disappear. Indeed, in overall mathematical reasoning, the boys made greater gains than the girls over the two years, although the girls' progress was greater on one subtest involving word problems. Although we had expected that the girls, once exposed to classroom instruction, might compensate for possible earlier differences in their play experience, this was not the case. We seemed to have discovered another instance of what many call the Matthew Effect ("the rich get richer").

Was our Saturday Club intervention effective? With regard to this question, the overall effects were statistically significantly positive when we looked at the strides made by the intervention group on the math-related measures compared with those made by the comparison group. The difference between the groups was not significant in terms of their growth on the verbal measures and only marginal on the visual-spatial measures, tending to confirm the notion that Saturday Clubs had facilitated the mathematical reasoning of the participants. There was also a closer relationship for the Saturday Club group between their mathematical reasoning and verbal reasoning, which we thought might be the result of the effort we had devoted to translating ideas from one kind of medium (e.g., talking, drawing, or writing) to another (e.g., building models). We were, of course, pleased with these findings. Indeed, it was these findings that gave us the courage to share with parents and teachers the philosophy underlying our approach and the methods we had used.

CHAPTER 3: Playing With Math

Appreciating, Not Teaching: How to Tune In and Enjoy Your Child's Mathematical Thinking

When the famous primatologist, Jane Goodall, was a little girl, she was fascinated by nature. At a very tender age she asked excellent questions—How do ants sleep? How do hens lay eggs? She showed keen powers of observation in her attempt to answer such questions. One day, at about the age of four, she decided to find out exactly how hens do lay eggs. She went into the chicken coop early in the morning and did not emerge until late that evening, by which time her mother, not knowing where she was, had become frantic. As she was about to call the police, little Jane came running out of the chicken coop exclaiming, "I found out! I found out!" Despite being caught up in emotions of relief and anger, her mother asked "What did you find out?" Only after hearing Jane's detailed description of egg-laying and sharing her excitement did her mother say, "Next time, just let me know where you are going to be so I won't worry."

Tuning In

Perhaps parents don't have to go to such extraordinary lengths to demonstrate that they, too, can tune into and support their child's curiosity. As a parent you are in a wonderful position to facilitate your child's mathematical thinking. You can do this first and foremost by relaxing. You do not have to teach your child anything! In fact, you can help your child most by declining the role of active "teaching" or drilling facts or imparting information—unless of course your child asks you to teach or show her a specific procedure. Instead, you get to watch and observe and play with your child and with mathematical ideas. Despite the great temptation to teach facts or procedures, especially with children who soak up math like a sponge, our experience with the Math Trek participants as well as with our own children, reveals that direct, didactic instruction, while it has its place, too frequently fails to achieve its aims.

One reason direct instruction fails is that a topic imposed on the child may not fit with the child's current issues. Children are incredibly sensitive to the issue of "whose agenda is this, anyway?" You have probably noticed this when simply playing a pretend game with your child; your ideas are sometimes not very welcome, are they? You have probably also noticed that when you simply let your child tell you what to do in the play, it goes quite smoothly. One trick, then, is to let your child give *you* a problem to solve.

"Oh, Cindy, it's going to take about 20 minutes to get to dance class. Do you want to give me a math problem to solve?" "Sure. Here's the problem, umm, okay, how 'bout 10 plus 10 plus 10 plus 3? That's easy! 33!"

You notice in this example that 5-year-old Cindy actually answered her own question, and did so with alacrity and pleasure. In posing this question herself, Cindy not only made the car ride less tedious, she also offered valuable clues to her Mom as to the parts of the number system and the operations on which she was currently working. In this case, Cindy was very interested in the fact that numbers can be decomposed in terms of tens and ones and the fact that tens are very easy to add together. Now Cindy's mother could either respond with a similar problem, or ask the following: "Wow. How did you solve that so quickly?"

By asking that question, she's letting Cindy know that she's interested in how Cindy *thinks* as well as in her answers. She's also giving Cindy an opportunity to *communicate* her *mathematical reasoning*. The communication of mathematical reasoning is an important skill in its own right. Communicating mathematical reasoning also contributes to deeper mathematical knowledge and gives the child a sense of mathematical power.

In asking questions about how a child solves a problem, and why she did it that way, we are also letting a child know that the process of mathematical reasoning is itself intrinsically interesting. Indirectly, we convey the idea, too, there are alternative ways to arrive at the same answer. By asking how and why questions we let children know that we take them seriously as mathematical thinkers. This is also a way to convey respect for a child's current level of mathematical development. In contrast, if we praise right answers and speed alone, we convey a belief that ability equals speed and accuracy, leading bright children to wonder if perhaps they are really dumb when they inevitably get a wrong answer or need to take some time to figure out a challenging problem.

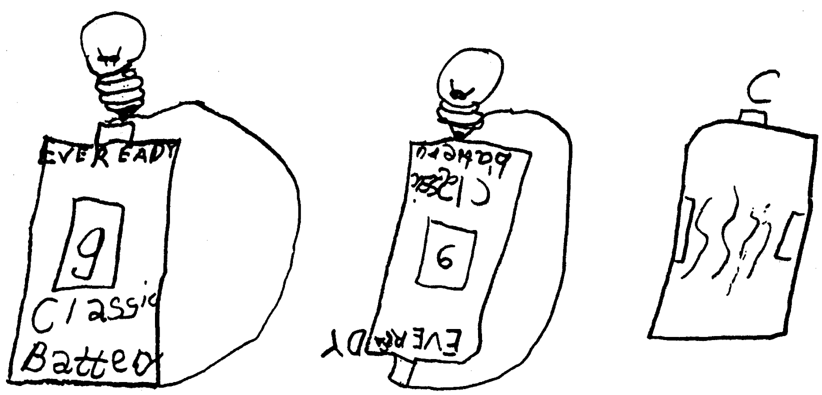
In a sense, then, parents would do well to have a bit of both Jane Goodall and her mother in them. By this we mean that for parents, their own children are far more fascinating to observe and understand than any hen laying eggs. By observing your child's mathematical play, you can both come to understand your child's current mathematical agenda and find a way to be nurturing and supportive of his or her mathematical intelligence.

So sit down on the floor with your child when she is playing with blocks or construction toys or putting a puzzle together or drawing a picture. These activities in themselves involve mathematical and visual-spatial intelligences and will let you know the skills and concepts on which your child is currently working. For instance, putting together the tracks of a wooden train set involves number (counting the number of different kinds of pieces), categorization (there are different kinds of tracks such as curved pieces, short straight pieces and long straight pieces), visual-spatial skills (figuring out how to make all the pieces fit into an oval, circle, intersection, or overpass), and the ability to keep both the wholes and the parts in mind at once. You will find that children set their own tasks and goals, spontaneously and instinctively setting new challenges for themselves (e.g., putting the track together all by themselves, changing a circular track to an oval one, making one track within another one) as well as repeating and rehearsing past pleasures and triumphs.

As children grow older, much of their intellectual play becomes more interior and less visible. School-age children still play with blocks, construction toys (e.g., Legos and models) and drawing, investing their constructions with more complexity. However, they also spend time playing other games and simply thinking. Board games such as Monopoly or Scrabble that involve judgment and strategizing require mathematical calculations and problem-solving and provide a great window onto children's imaginative reasoning abilities. For instance, Michael loved Monopoly and played it incessantly. In order to make the game even more interesting he would double the value of all the properties, and, to keep things in proportion, doubled the rent and prices of hotels as well. These calculations, in addition to the strategies the game already called for, kept him quite involved and busy.

Not By Words Alone

Another characteristic of mathematically talented children is that they pick up words very quickly, often astounding their parents with their wide-ranging vocabulary. However, uttering big words does not guarantee their comprehension, which frequently lags behind word acquisition. Part of what young children attempt to do by using a big word or term is to figure out the right context and meanings for that word. This issue was brought home by one child at Math Trek who possessed an impressive vocabulary of electricity terms, yet couldn't figure out how to get a bulb to light using a battery and wire! So listen carefully to how your child is using the word and find some way for your child to enact her mathematical knowledge as well as to verbalize it.



Mathematizing the World: Finding Mathematical Possibilities Everywhere

One way that parents can encourage their children's mathematical thinking is to help them find mathematical possibilities everywhere. This is not a hard task; mathematics can be thought of as a lens through which to see the world as well as a language to describe that world. Looking through that lens reveals patterns, systems, connections, and relationships. Anywhere you look, you are surrounded by a world composed of shapes, angles, and numerosity. Math also provides a way of quantifying time and space so we can think more precisely in historical, geographical, and even astronomical ways.

Indeed, one interesting aspect of development is that while even very young children notice and appreciate shape, number, time and space, their first understandings of these concepts are qualitative, not quantitative in nature (Elkind, 1976). For instance, young children think quite impressionistically about time: *Yesterday* can mean "anytime in the past" and *tomorrow* therefore means "anytime in the future." For the young child, any distance is a very long way, as even a short car drive will quickly prove. It is only with more development and experience that children begin to forge quantitative and systematic understandings of those concepts. Even very precocious children go through this developmental process (although more quickly than their peers) with its attendant overgeneralizations and misconceptions, some of which are quite comical when viewed from a more sophisticated perspective.

During the last session of the first year of Math Trek, the teacher personally said good-bye to each participant. When it was time to say good-bye to Sophie, the teacher said, "Well, when you come back in the fall we'll see how much you've grown." Sophie responded seriously: "Oh, no, I won't grow any over the summer. I just turned six so I did all my growing for the year. I won't grow again until I turn seven."

These naive theories about how things work are charming to listen to and can often serve as wonderful conversation starters, with the parent simply asking more questions to draw out the child's theory. Paley in *Wally's Stories* (1981) describes many such conversations.

For young children, mathematizing the world merely requires pointing out a phenomenon and seeing if it takes. For instance, think about when you introduced shapes or colors to your child. Remember the excitement as he or she found rectangles everywhere? Or pointed out, "Dat's blue, so's dat and dat and dat!" The same can be said for more complex structures such as spirals, which occur in pine cones, snails, coiled garden hoses, and so on. On a walk, then, ask your child to find examples of spirals. It's amazing how a phenomenon will pop up once we are alerted to its presence. Spirals also lend themselves to art projects. A great activity for that pre-dinner witching hour is to ask your child to cut out a variety of big shapes from construction paper. Good choices are triangles, squares, and circles. Then ask your child to construct spirals from each shape. Can it be done? What makes each one a spiral? They look very pleasing taped on the ceiling.

Other mathematical features to look for in our everyday environment are particular shapes such as rectangles or squares. This shape hunt could lead to an interesting discussion of your child's definition of a rectangle. For instance, is a square a rectangle? Why or why not? What about parallelograms without right angles? Is that skinny, elongated rectangle over there considered to be a rectangle by your child? One of the fascinating things about children's thinking is that it is often qualitatively different from our own. Yet all children are engaged in making sense out of their worlds and to do so they are constantly theorizing and conjecturing. So children have a concept of rectangles and a theory about what counts or doesn't count as an example of one. This makes for interesting discussion.

Parents are generally unsuccessful in changing their youngsters' minds by direct argument. You can suggest counter-examples or tell your point of view, but simply imposing your adult understandings on your child rarely works. Think about the grammatical mistakes your child made when learning to speak. All the corrections in the world will not get a child to construct grammatical speech faster, and, in fact, may inhibit the child's fluency. And yet your child ended up speaking in a perfectly grammatical way. Development is a glorious thing, indeed. Children, in their own good time, and through interactions with a world of stimulating objects and caring others, will sort out their concepts and definitions of things. Going back to the issue of rectangles, it may well be that young children form a prototype in their minds of what rectangles should look like. This prototype probably looks something like this:

Rectangles that are distinctly different from this prototype, such as tall skinny ones, or tilted ones or even squares seem too far removed from the prototype to qualify as a rectangle. However, as children develop intellectually, they will expand their conception of rectangles to include these outliers. Simply raising the issue of what is or is not a rectangle invites children to reflect and to theorize.

Number Play

Everyday life with a young child is full of opportunities to engage in number play. By number play we mean creating word problems or computational problems that involve all the situations that children naturally encounter. Let's take birthdays and ages for an example.

Justin's dad had just turned 38 and his mom recently turned 32. Justin's mom playfully asked her son the following questions over a two week period: How long until your dad turns 40? Justin quickly exclaimed, "In two years! Easy—It's like 8 and 2 are 10. Forty is like 30 and 10." His Mom responded with the following question: "So how *long until I turn forty?" Justin paused for a while and then an impish smile played on his mouth: "Eight. It has to be 8 because 8 and 2 are 10 so 2 and 8 have to be 10, too."*

The next day Justin's mom asked, "So how old will you be when I'm twice as old as you?" Justin did not have a ready answer. In fact, he frowned and went back to his Lego construction. Justin's mom wisely let the matter drop, figuring that problem was simply too hard for Justin. However, two days later, Justin announced his answer. "Mom, I think I got it. I'd be the same age you were when you had me! So I would be 26 and you would be 52." Justin's mom was pleasantly surprised that her son had worked out the problem.

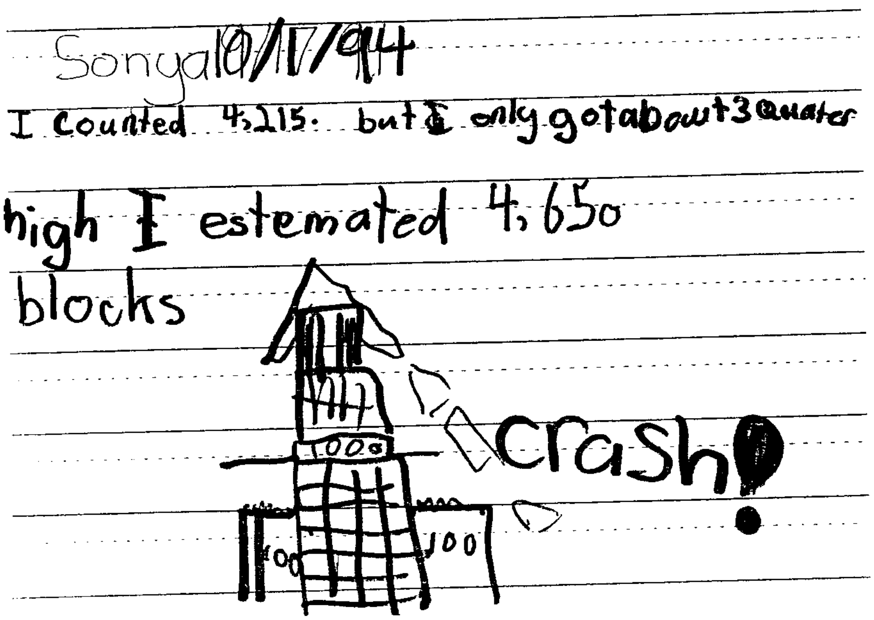
A variation on the age problems might be to ask your child to categorize the various ages in the house as to whether they are currently an odd number or an even number. Odds and evens hold a particular fascination for bright young children; this way of categorizing numbers seems to be a first introduction to number theory and the interesting proposition that not all numbers are the same; they have different properties and features. Taking a walk down any street lined with homes affords the opportunity to determine which addresses are odd and which are even. In big cities with 4 or 5 digit addresses, the task can be challenging, thoroughly testing a child's knowledge of odd and even numbers.

Another problem concerning ages of household members is to ask what two numbers when combined total a person's age. This task is relatively easy if a person is six; how about if a person is 7? This problem allows a child to enter into more number theory issues, such as the fact that even numbers divide equally into other whole numbers while odd numbers do not. It is also a great invitation to consider the reciprocal relationship of doubling and halving. And you can always extend the problem to the parents or grandparents' ages. For instance, what two numbers combine to make 38? Children love to play with double numbers, and so these problems seem to be innately satisfying for young mathematical minds.

Playing with proportionally sized blocks is another inherently mathematical activity that affords plenty of opportunity for informal mathematical play:

Josh was sitting on the floor making an elaborate house for the Grand Vizier (he loved the video, Aladdin*). His mom came in to his room, sat down on the other side of his structure, and simply watched him build. Every now and then she would utter an appreciative "wow" and sometimes Josh would instruct her to place a block on her side of the building. She also commented on how symmetrical the construction was. Josh was pleased that she had noticed that: "Yeah, I like it when it looks balanced." After a while, Josh ran out of the long blocks he liked to use. He talked to himself as he solved this problem: "Okay, I don't have any more of these, but I can use these little blocks instead. I bet I need four of them to make one of these long ones. His mom said, "Can you prove that?" So Josh laid out one of the long ones and lined up the four shorter ones to prove his point. The building was set aside while both Josh and his mom entered into the spirit of the game by posing more proportional problems. By the time Josh got around to* *determining how many of the very shortest blocks would equal the very longest block he didn't even need to prove it physically. Instead, he said, "It's gotta be sixteen." His mom asked, "How did you know that?" Josh immediately replied, " Because. It took eight of these blocks to make one long one, and it took two of these little ones to make this medium one. Wow! It goes in order! You just double the number!" Josh's mom said, "Let's say all the numbers we found." Together, they chanted, "Two, four, eight, sixteen!"*

This playful interlude demonstrates how easy it is to weave a little math into a play context in a way that still allows the child to be in charge.



Mathematical Mistakes: A Clue to Reasoning

> *In her book,* Peripheral Visions, *Bateson describes the responses of two famous visitors to an optical illusion experiment in Princeton. When Eisenhower looked into a box that distorted perspective, he attempted to touch different points in the box with a stick. When he could not do so, he became quite angry and threw down his stick. When Einstein explored the same box, he became intrigued with the errors he made and used them to further explore the box.*
>
> Bateson (1994, pp. 73-74)

Some children love to play with math in traditional ways, by embracing number facts, computations, and even workbooks. These children are clearly attempting to master the number system, with its regularities and relationships, and finding the process of completing row after row of computation to be very satisfying. These children often quickly absorb the written procedures (algorithms) for the different operations (e.g., regrouping in addition and subtraction) before they actually understand the rationale for those procedures (i.e., the '1' that is carried in addition is really a ten), paving the way for the typical errors that math educators call "bugs" (Brown & VanLehn, 1982; Ginsburg, 1989).

Nina worked on her math workbook for almost an hour one rainy afternoon. She proudly showed her dad the section she had just completed and asked him to mark all the ones she got right. Nina liked the sight of all those check marks on her worksheets. Her dad sat down with the marking pen and went over the subtraction problems. This section started out with one-digit subtraction, moved on to two-digits minus one-digit, progressed to two-digits take away two-digits without regrouping and ended with regrouping problems. Nina got everything right until the regrouping section. Her dad started to simply mark them wrong and then became curious as to why Nina had made those mistakes. He studied them for a bit and found a pattern to her mistakes; they were not careless mistakes. He figured out that Nina always took away the smaller number from the larger number no matter where the smaller number was located (e.g., for 83 - 27 her answer was 64 because she took the 3 away from the 7). He called Nina over and asked her to explain how she got the answer. She very succinctly said, "You can't take a bigger number away from a smaller number; what kind of sense does that make?" Her dad pointed out that she had subtracted six from 13; how did she do that? Nina replied, "For that one I just counted backwards and got seven." He then asked, "But isn't 83 bigger than 27?" Nina looked confused for a minute and then said, "Yeah, but three is not bigger than seven!" With that she grabbed the workbook and went off to play a computer game.

The "always take the smaller number away from the larger number" bug is a common one that makes sense in light of the fact that subtraction is often taught in the incremental ways organized in the workbook.

For a little variety, try some mental math by verbally posing a computation problem and having your child work it out without benefit of paper and pencil. Mental math prods children to find ways to take numbers apart (decompose) and put them together again (recompose) in flexible and useful ways. These invented strategies are often creative and helpful in mastering the number system. The process of decomposing and recomposing numbers is at the heart of all computational algorithms, but in algorithms this process is hidden by the place value of the digits (remember that little '1' that is so often "carried" and "borrowed"?) and children often don't realize what is really happening in the algorithms even though they can master their use. Inventing their own strategies gives children a feel for what might be going on in a standard algorithm.

Anna had worked through her book of mazes and was getting restless on her family's trip to a zoo in a nearby city. Her mom asked her to figure out the following problem: If it's a 52 mile trip to the zoo and we've already driven 27 miles, how many more miles do we have to drive? Anna grumbled, "That's too hard. I need paper and pencil and I just dropped my pencil and I can't find it." Her mom said, "Try and figure it out anyway. You didn't drop your mind, you know." Anna replied, "No I can't; that's a takeaway problem and I'll have to borrow 'cause it would be two take away seven and I *can't do that in my head." Her mom once again urged her to find another way to do it, a way where she wouldn't need to cross out. Anna spoke out loud as she tried to figure: "Okay, we've gone 27 miles, so we need to go three more miles to get to 30. So that's three. Then ten more would be 40 and ten more would be 50. So that's 20, and don't forget that three, that's 23 . . . . How many miles all together did you say Mom? Oh, 52. Okay, that's just two more, so 23 and 2 is 25! We'll be there in 25 more miles. That's going to take too long." Her mom then asked, "So have we gone more than or less than halfway? Anna said, "More, 'cause 27 is more than 25. Okay, we're more than halfway there."*

Anna's strategy for solving the problem, after her initial struggle and reliance on the regrouping algorithm, involved adding on ones to get to a decade number, then adding tens on to get to the decade in question (50) and then adding on the remaining ones. The final answer was achieved by adding the tens to the ones. Her strategy showed that she wasn't locked into just one way of solving a problem, that she could think of the problem as a missing addend instead of a take-away problem, and that she could take numbers apart in terms of tens and ones. Being able to take numbers apart in terms of tens and ones is the basis for understanding place value.

Mathematics and Stories

Richard Feynman, in one of his autobiographies (1985), remembers his father reading him stories about dinosaurs. When his father read about a dinosaur that was 25 feet high, for example, he would stop and ask Richard how tall that might be—one story high or two stories, or as tall as the chimney? Given that stories and factual books are filled with mathematical concepts and measurements, pausing to consider what it all means, and trying to find some everyday markers to help with that sense-making, is a valuable and enjoyable activity. It also subtly teaches a child that there is a way to make sense of what one reads, and that sense-making is available to a child through his or her own experiences and observations. Feynman's father knew that you always use what a child already knows, his current frame of reference as a stepping stone to further knowledge.

There are numerous story and nonfiction books available these days that are designed to support children's mathematical explorations. A partial list appears in Appendix A. Remember, though, not to read too fast! Leave time for thinking and talking! If this leads you and your child to connect to ideas outside the story, so much the better. The story can be completed another day.

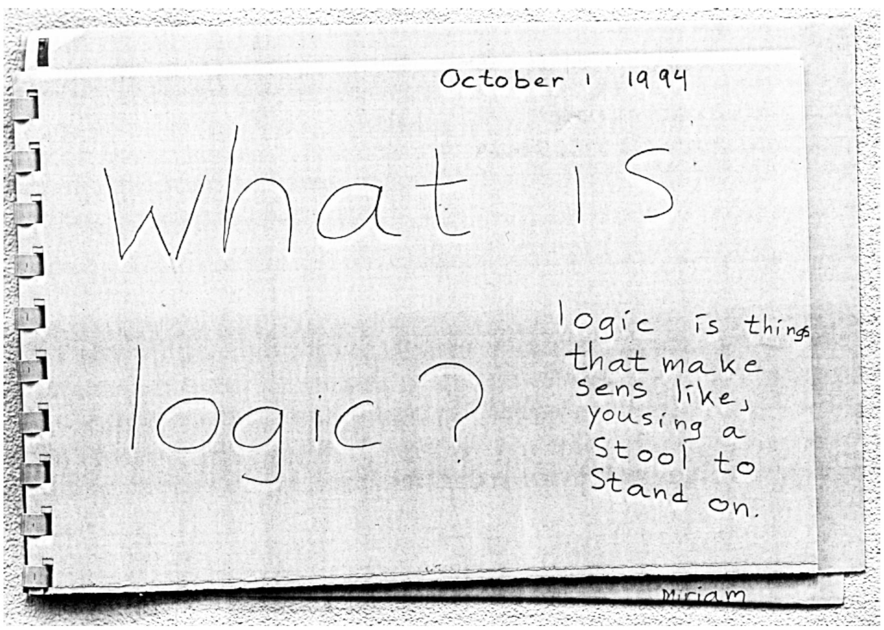
Mathematics and Writing

Peter, one of the Math Trek participants, hated writing. The worst parts of first grade for him were all the requests to write. His mom was puzzled by Peter's dislike of writing for he loved to read and draw. His favorite subject, however, was math. During one of the second year Math Trek sessions, the children were asked to make a drawing and write a story that would make sense of some simple equations. One equation was

0 - 3 = -3. Peter loved negative numbers and was intrigued by the challenge of coming up with a plausible story. He spent a long time drawing a picture of a man digging in order to get to some underground pipes. He then wrote a comical story about a man who had to dig three levels underground in order to get to a certain pipe.

After this episode, Peter would often choose to write a story about an equation instead of other activities offered. Connecting writing to a subject he cared about gave him a reason to write. While other children may not share Peter's initial feelings about writing, writing about math is an excellent way for children to make sense of equations and to invite them to reflect on math.

Many children like to keep journals and we exploited this at Math Trek by asking the children to record their activities in one of three books: My Book of Numbers, My Book of Shapes, and My Book of Logic. The children could choose which book they thought most appropriate for the activity they were recording, and they were allowed to draw or write. Recording and representing one's thoughts is another kind of problem-solving as well as an expressive activity.

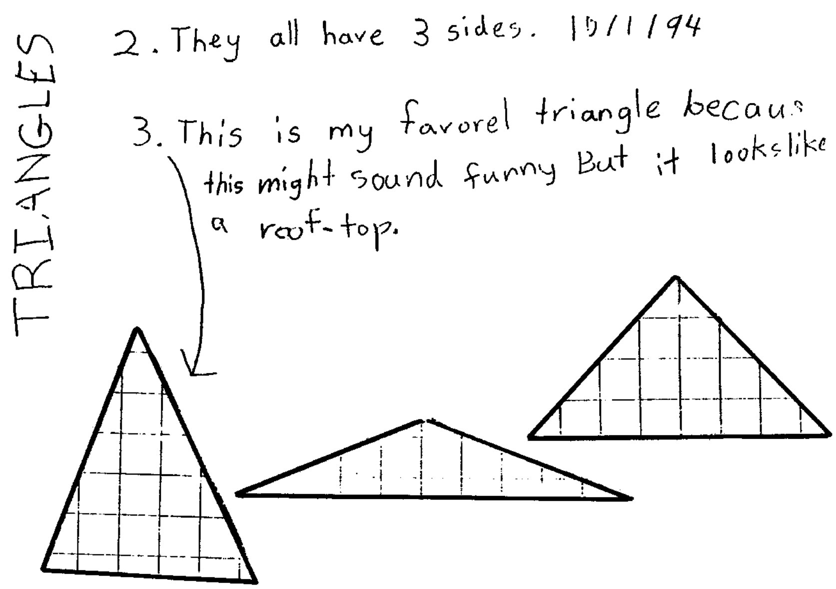


Mathematics and Art

Drawing, as mentioned above, is another way of representing one's thinking, one that requires problem-solving in the visual-spatial domain. Children (and their parents) can use drawing to either as the means to solve the problem or as a way to show a solution.

Kenny got right to work on a problem assigned at Math Trek: If there are 64 lights on each of 12 Christmas trees, how many lights are there altogether? Kenny first *drew the 12 trees. Then he drew, on the left side of each tree, six big lights and, on the right side of each tree he drew four little lights. Inside each of the big lights he wrote the number '10' to remind himself that each of the big lights symbolized ten lights. Then he proceeded to count all the tens. When he finished counting the tens he wrote his count of 720 down and went on to count the remaining lights by fours. He wrote down 48 for the ones underneath the 720 for the tens and added them up: 768. In this way, Kenny figured out his own procedure for multiplying 12 x 64.*

It's clear to see in this example that Kenny used his drawing to figure out the answer. His use of a big light to symbolize ten little lights was a particularly clever use of the medium. In general, asking children to use drawing to solve problems is an enjoyable way to help them see that visualizing a problem can lead to its solution. Visualizing problems is a skill called upon more and more in higher mathematics and science. And the seeds can be planted in preschool.



Mathematics and Science

From astronomy to the chemistry of cooking, children's early appreciation of science requires the use of math to quantify, compare, and classify. Experience with the uses of mathematics in science, and the many scientific contexts that use math as a tool, will help children create an internal number line, from negative numbers (say, a centigrade thermometer) through the smallest of fractions (say, 1/8 of a teaspoon) to the headiest of large numbers used to describe the space between our planet and a distant galaxy. Math, in the context of science, illustrates the interplay of processes such as measurement, estimation, and calculation, and the necessity for a logical system of written representation that can express the very small and the very large as well as how to arrive at those quantities. Thus, scientific activities, experiments, and thinking can help children see that aspects of math that are usually studied in isolation all have a place and a purpose.

While even some very math-talented children are mystified by conventional ways of representing numbers, other children are fascinated by the logic that underlies numerical representation, for example, using positive exponents and powers to describe astronomical distances and negative exponents and powers to describe atomic weights. Astronomy and chemistry provide two incentives for grappling with these kinds of numbers and their written expression. Considering math and astronomy together, or math and chemistry together, may well help the children who are mystified as well as intrigue the children who easily grasp the logic.

Science experiments that are easy to perform in kitchens and bathrooms are detailed in many books, some of which are described in the Annotated Bibliography at the end of this book. We also recommend having a microscope and a telescope available (many inexpensive yet quite good ones are on the market) as well as a magnifying glass. Rocks, shells, seed pods, dirt, drops of water, even tiny toys make good subjects for study and discussion. Provide a little notebook in which the child can record data, observations, and drawings that render the structure of what is seen. The notebook might even be the perfect place to write a poem, which is simply another way to record observations and express the structure (and meaning) of what is observed.

Parents can also invite children to ponder the philosophical and poetic issues involved in contemplating the minute and the immense. Children love to imagine a whole universe within an atom or to imagine that our whole world is really just a microscopic speck on some giant's skin.

Children, with their immense curiosity and need to know are often the best source of questions and conversation-starters. Indeed, listening to children's observations and questions while taking walks, cooking, gardening, or merely gazing up at the moon is the perfect place to begin doing science.

Vicky was helping her mom make some potato pancakes. The recipe called for egg whites whipped into stiff and foamy peaks. Vicky competently separated the eggs into two cups when her mom, who was busy grating potatoes, suggested using a bigger bowl for beating the egg whites. Vicky began to scowl (she hated to think she was doing something "wrong" or needed to be told what to do). Her mom quickly changed gears and suggested that they perform an experiment: "Let's experiment by whipping some egg whites in a little cup and some egg whites in a bigger bowl and see which way is easier." Vicky liked that suggestion and quickly set up her experiment. She was very intrigued to find that it was indeed easier to whip the egg whites in the bigger bowl. "So why does that work, mom?" Her mom, distracted now by grating onions, asked her to answer her own question. Vicky ventured a guess, "'Cause there's more room? But wait, how come beating them in any bowl makes them get so big? Yolks aren't like that. What gets added when you beat them?" Pretty soon Vicky and her mom were discussing volume and air and measuring the snowy white peaks of beaten eggs.

CHAPTER 4: The Power of Big Ideas

Love, death, the cruelty of power, and time's curve past the stars are what children want to look at.

Bly (1981, p. 175)

For many children there are other powerful ideas that they would gladly add to the above list. Some of these powerful ideas are mathematical in nature and provide a great way to infuse liveliness into the mathematical conversations you have with your child. These powerful ideas are also a way to convey that there is a transcendent aspect to mathematics, something deeper and larger than calculations, something mysterious that leads to more and more questions.

Infinity

Cindy and her mom were walking home from the park one afternoon when Cindy proudly announced that she could count up to a hundred and, without any prompting, proceeded to do so. Given the fact that they still were about 10 blocks from home, her mom idly asked, "I wonder what number comes after a hundred?" Cindy skipped around her mom and said, "That's easy; a thousand! Then a million! Then a billion! Then a zillion!" Her mom then asked, "I wonder if there are any numbers after a zillion or if the numbers stop there." Cindy emphatically replied, "'Course not! That's it. Zillion is the very end." Cindy's mom then asked, "What answer do you get if you add 1 to a hundred?" Cindy did a pirouette and said, "Easy! A hundred and one!" Cindy's mom kept the game going by asking, "What answer do you get if you add one to a thousand?" Cindy thought for a second and said, "A thousand and one!" Catching on to the pattern, she exclaimed, "So if you add one to a million you'd get a million and one! Then it's a billion and one!" Cindy stopped walking for a minute and thought and thought. Finally, she started walking again, this time very slowly, and soberly said, "Well, if you can go a hundred and one, a thousand and one, and a million and one maybe you could go a zillion and one, too. That means the numbers never stop, 'cause you can always go one more."

Infinity is definitely one of those powerful Big Ideas. Once your child, perhaps with your prompting, discovers the concept of infinity, there are many ways to explore the various meanings and implications of the concept. Indeed, infinity is a concept that has a powerful affective pull on young children; to understand the concept deeply is to have your understanding of the world shaken up. A comical example of this occurs in the first scene of the Woody Allen film, *Annie Hall*, when the very young Woody Allen character has discovered that the universe is infinitely expanding. He is so upset by the idea that he ceases all work. For if the universe is constantly expanding, what's the point?

Astronomy is one context in which to explore the concept of infinity, especially on a clear and starry night. Given the latest information from the Hubble telescope, infinity becomes an even more impressive concept. Infinity, in the sense of vastness, is also experienced at the ocean or at the desert, especially when filtered through questions such as: "How many grains of sand do you think there are in the world?" Staying closer to home, several plastic mirrors set up at a particular angle will give children a visceral experience of reflections going on forever.

In addition to the infinitely vast, there is the infinitely small, too. Ask your child to get to the door of a room by taking steps half the distance of the previous step. Your child will notice that while she is getting closer to the door, she will never actually reach the door.

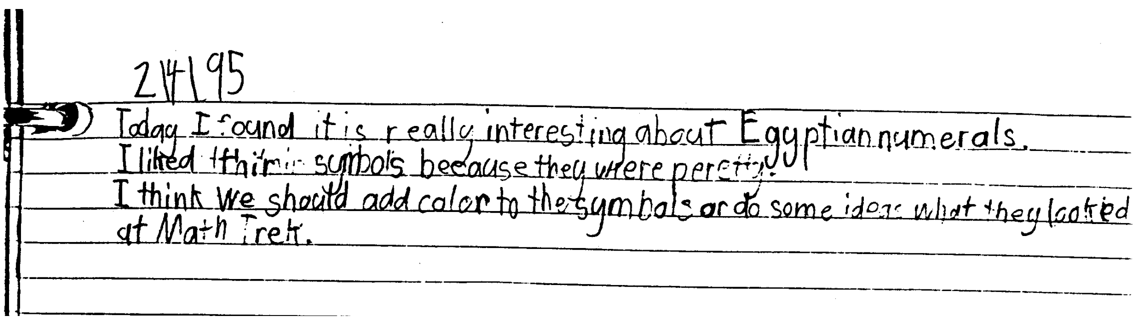
Zero

The concept of *zero* is another big idea that intrigues children and invites a more abstract and algebraic approach to math (Wellman & Miller, 1986). Zero is both a place-holder and a symbol for the null set, i.e., nothing. It is also the dividing point for positive and negative numbers (or, as one child expressed it, "Zero is the river between the forest of negative numbers and the meadow of positive numbers.") The concept of zero and the numerical symbol for it were relatively late mathematical inventions (approximately 600 A. D.) and therefore of great historical interest. Young children are fascinated when they find out that not all numerical systems possessed a symbol for zero; they wonder how that could be, and, more importantly, how zero was invented. They also wonder how calculations were dealt with that resulted in zero. Leading children to think about mathematics as a human invention inspires them to think of themselves as creative and inventive.

Parents can capitalize on their children's curiosity about zero by posing word problems and calculations that involve zero. For instance, a series of questions about adding, subtracting, and multiplying with zero helps to get a child thinking about the special role of zero in various operations. You could also invite children to think up story problems for equations in different operations, such as $3 - 0 = 3$, $0 + 3 = 3$, $3 \times 0 = 0$ and 3 divided by 0. Follow up with $0 - 3 = -3$. You might also ask your child what zero means in $1.00? In 105? Exploration of zero across all these cases will get your child thinking and comparing. Also ask your child to draw a picture to go with her story. The pictures often capture how a child really conceptualizes operations.

Numeration Systems

Other *numeration systems* also hold a fascination for young mathematical minds. Again, learning that other cultures, from ancient times on, invented numeration systems leads children to reflect on mathematics as a cultural invention that can take many different forms. Young children are intrigued to discover that other numeration systems may use different bases and very different symbols than our Hindu-Arabic system. In a sense, learning a different numeration system is like learning a new language; struggling with a different structure throws into relief the structure of our own native language. So ask your child not only to learn about Roman numerals or Egyptian numerals, but to compare those systems to what they use. The children at Math Trek were amazed that the Egyptian system does not possess place value; they realized that calculating in a system that does not possess place value is a cumbersome process. The excellent little book, *Counting in Greek* (Draze, 1990, see Appendix A), is a good place to start. Once your child has mastered a different numeration system, ask her to calculate using that system. You might even suggest that your child invent her own system!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Today I found it is really interesting about Egyptian numerals.
I liked their symbols because they were pretty.
I think we should add color to the symbols or do some ideas what they looked at Math Trek.

Reversibility

Another big idea in mathematics is *reversibility*, a process which takes many forms and which results from the logical structure of relationships that comprises mathematics (e.g., subtraction is the inverse of addition, division is the inverse of multiplication, and an equation can lead to a graph or vice versa). Thus, one way to challenge children and encourage flexibility and alternative ways of examining things is to ask children to go in reverse (e.g., the Chip-Trading Game in Appendix B), or to provide them with the answer and have them tell you the question as in, "If 14 is the answer, what is the question?" Or, as suggested above, to give children an equation and ask them to create the story that would fit that equation.

Indeed, one of the big ideas in life is that not everything *is* reversible. Children's cartoons suggest, for example, that the Road Runner can die and come to life indefinitely, over and over. Can grownups turn into children again? Children come gradually to understand *cycles* as different from *reversibility*, sometimes in painful ways from which we cannot fully protect them, but we can help them by being sensitive to the nature of their discoveries and offering comfort for their uncertainty.

Equivalence

The idea that ten ones equal one ten and, conversely, that one ten equals ten ones seems quite obvious to grown ups. However, the idea of *equivalence* is not at all obvious to young children. Equivalence is a concept that children need to construct over time and in many different contexts. Equivalence pervades all of mathematics, from place value to equivalent fractions to geometric congruency, and therefore is well worth children's intellectual energy. A particularly enjoyable context in which to start thinking about equivalence is money.

Janie's dad had a habit of throwing all his spare change into a big dish on his dresser. About once a year, when it became very full, he decided to do something about it. This year, he enlisted Janie's help by asking her to sort, count, and tally the amount of money he had collected. When he put the big dish of coins onto the kitchen table Janie's eyes grew very large. He started by asking Janie what she knew about coins. Janie immediately said, "I know ten pennies are the same as a dime! And, a nickel is five pennies." Janie's dad asked, "And how many nickels are in a quarter?" Janie put two nickels together and said, "Okay, five and five are ten, and another five and five are ten, and four of them would be ten and ten, so that makes twenty, and one more would make 25. So five nickels is the same as a quarter. And I already know that four quarters make a dollar." Janie's dad announced that he thought she was ready to begin. It took her a very absorbing half hour or so, but she put all the pennies into groups of tens, all the nickels into groups of fives, and all the quarters into groups of fours. Then she got a piece of paper and a pencil, figured out how much money each group represented and put those numbers down, taking time to convert into dollars and leftover cents. She made a few counting mistakes, but it was clear that she could use the idea of equivalence to make sense of money and help her dad with his yearly task.

It is also by virtue of equivalence that the base system works; without equivalences, we would be left with tally marks. Young children take base-10 for granted—as, indeed, do grownups—but are fascinated to learn about different bases. Again, exploring different numeration systems involves learning about different bases (e.g., the Babylonian system) as can examining analog clocks (another way in which we use equivalences). There is also a wonderful game that affords play with different bases and therefore with equivalence: Chip-Trading. This is a game that is easy to make and play at home, and is described in Appendix B. A good base to start is base 3 (or, as it is called in the game, "The Land of Threes").

Cooking is also a natural and compelling context in which to explore equivalence, given our system of teaspoons and tablespoons, cups, pints and quarts, etc. Cutting recipes in half or doubling or tripling them provide natural problems to solve that demand an understanding of equivalence.

Visual Representations of Number

One wonderful feature of math is that it can describe, through equations, sequences, and functions, aspects of our physical world. Conversely, numbers can be the basis for pictures, as in data graphs. Playing with this visual aspect of math can be a source of wonder for young children, too, connecting them to the power of math as a descriptive language. Having some graph paper on hand can be an easy way to allow your child to begin to play with visual representations of number. (If you can, find paper that is 1 cm squares, rather than being blocked off into smaller squares.)

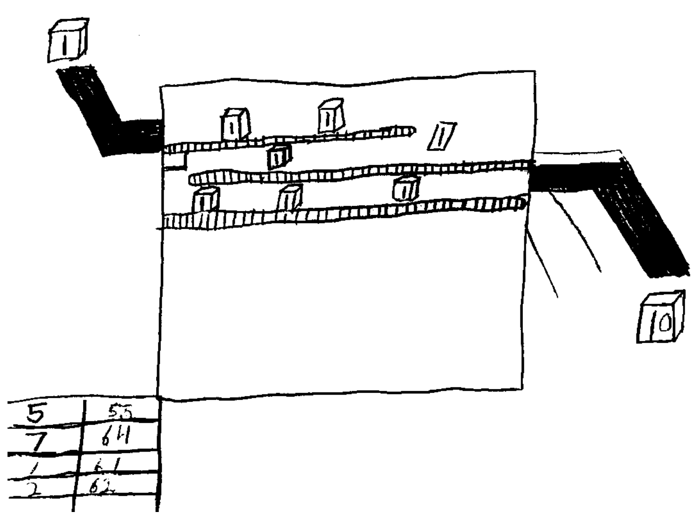
A compelling game for young children is to graph their names by assigning a number from one to nine to each letter of the alphabet:

1	2	3	4	5	6	7	8	9
A	B	C	D	E	F	G	H	I
J	K	L	M	N	O	P	Q	R
S	T	U	V	W	X	Y	Z	

Going in a clockwise direction, children can graph each letter of their name and attempt to return to their starting point on the graph paper. For example, RANDY would be a line of 9 squares; turn right for a one-square line; turn right for a five-square line, etc. Intriguing patterns emerge from the graphing, pleasing children and their parents. Children can go on to graph their parents', siblings', and teachers' names, too, and to compare the graphs that emerge from different names. How do short names compare to long names? What do the graphs look like when a name repeats letters or letters with the same numerical values? Why do some name graphs return to the starting point while others do not?

A monetary value can also be assigned to each letter of the alphabet; children can then figure out how much their names are "worth." Doing both activities gives them two new ways to appreciate their names, one visual, the other numeric.

Another way to begin play with the visualization of number is through imaginary magic number machines (see Appendix B for description of this game). This game uses simple functions in a "Guess My Rule" format. For instance, when the number two is put into the magic number machine and a 10 comes out, then a three is put in and a 15 comes out, the rule must be "multiply by five." A graph with X and Y axis can be drawn, and the numbers that are put in and the numbers that come out (e.g., 2 and 10, 3 and 15) become coordinates that can be graphed as points. Line segments, through the points on the graph, can then be drawn. This game provides a simple but elegant introduction to coordinate graphing and enables the children to see that they can predict, from the graph, number relationships that they had not yet graphed ("Cool!" was Maggie's word for this discovery). Children enjoy making up the rules for parents to guess as well as the reverse. As children become more experienced with this kind of graphing, squared and cubed functions can be introduced as well as negative numbers.



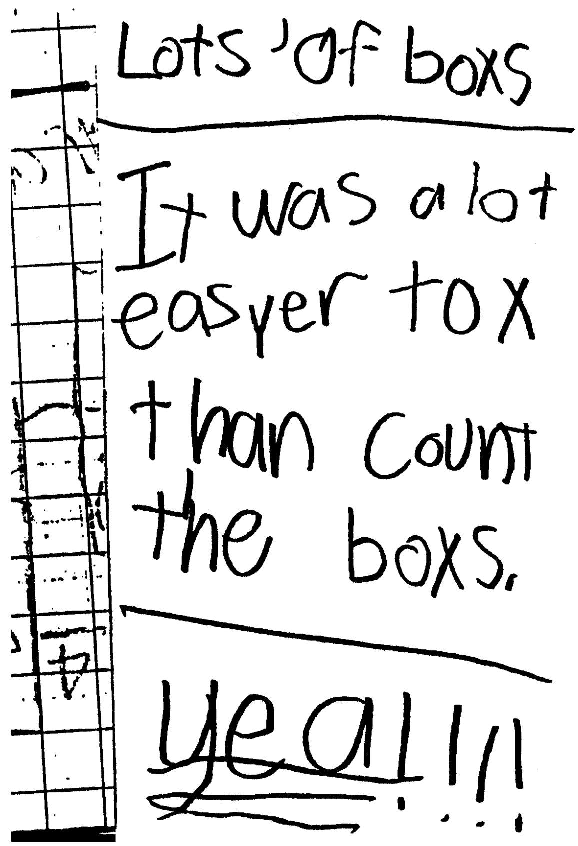
Numerical Representations of Shape

Reversibility in mathematics takes yet another form; there exists not only the visual representation of number, but the numerical representation of shape. One can start with either the visual or physical representation and attempt to get to the other.

Janie enjoyed drawing shapes. One afternoon she started with triangles and moved on to rectangles, pentagons, and hexagons. Under each shape she wrote the number of sides. After the hexagon she decided to keep on going and drew several more shapes, each one a side more than the last. As she drew a 12-sided shape she exclaimed: "It's getting close to a circle! Twelve sides is close to a circle."

In this example, Janie experienced a correlation between the number of sides and approaching circularity and one way in which a number (in this example, the number of sides) can represent a shape. Janie would very much enjoy a book by Burns (1994) that deals with the same theme, *The Greedy Triangle* (see Annotated Bibliography).

Janie would also enjoy a game that is easy to play at home called "Lots of Boxes," from Kaye's book, *Games for Math* (1987). This game requires plain, 1 cm graph paper, and a die. The first throw of a die tells how many lines to go across on the graph paper, the second throw tells how many lines to go down. Using those two pieces of information, the player then finishes the box. Next, the player figures out how many little squares comprise the box. This game potentially can lead to an interesting exploration of the relationship of squares to rectangles as well as to an exploration of area and perimeter, two ways of numerically representing space. Children also have the opportunity here to invent strategies to figure out the area of the boxes they draw. And parents have the opportunity to ask their children to verbalize their strategies. This game may well motivate children to invent their own ways to multiply as they will soon tire of counting little squares. Through playing this game, children may also come to appreciate square numbers, and the numerical differences between successive squares, especially if a parent asks how many more little squares are added to get to the next bigger square. This game provides a good example of an *open-ended activity* (see also Chapter 6), for it allows play on many different levels, including some sophisticated ones (try using two dice and multiplying the results in order to determine the length and width of the boxes, or figuring out how to make a three-dimensional box!).



Different Kinds of Numbers: Negative Numbers and Fractions

> *. . . The number system is like the human life. First you have the natural numbers. The ones that are whole and positive. The numbers of a small child. But human consciousness expands. The child develops a sense of longing, and do you know what the mathematical expression is for longing? The negative numbers. The formalization of the feeling that you are missing something. And human consciousness expands and grows even more, and the child discovers the in between spaces. Between stones, between pieces of moss on the stones, between people. And between numbers. And do you know what that leads to? It leads to fractions. . . .*
>
> Hoeg (1993, p. 112)

The discovery that there are fractions and negative numbers is a source of wonderment and intrigue for young children. Fractions, in the form of halves and quarters, are noticed fairly early in life, given the necessity to sometimes share cookies or the limitations of small appetites. At first, young children assume that anything broken in two is considered a half; the important criterion of equality is not reckoned with for awhile.

The current bedtime story in Andy's family was Alice in Wonderland. *After reading the first chapter when Alice is falling down the rabbit hole, Andy yawned and said, "I think probably that's where the negative numbers must live, down in that rabbit hole." "Yeah," said his older brother Tim, "And the opening to the rabbit hole is zero!"*

This playful imagery reveals how strange as well as logical the negative numbers must seem to young minds. The search for an appropriate image or metaphor also reveals how determined some math talented children are to make sense of some aspect of mathematics. Parents can help in this endeavor by suggesting various models for negative numbers such as vertical or horizontal number lines and by posing story problems that yield negative numbers. For example, "James entered a curious building and first went up to the fourth floor where he thought his doctor was. When he got there, the receptionist told him to go down seven floors. James decided to get some exercise and walk down. On the way down, he tried to figure out what to call the floor where his doctor was. What do you think it should be called?"

Again, giving children an equation and asking them to come up with the story is another challenging example of mathematical reversibility that will stretch young minds to come up with a reasonable context for a calculation that yields a negative number. For children advanced enough, all the permutations of operations utilizing negative numbers could be explored (e.g., multiplication of negative integers, of a negative times a positive integer, subtraction of two negative numbers). Creating reasonable stories for how a negative integer times a negative integer yields a positive integer would be a creative feat, indeed.

Simple paper-folding is an elegant way to learn more about fractions as smaller and smaller fractions are created with each successive fold of the paper. Cutting out shapes in the folded paper and then opening up the whole piece of paper yields delightful patterns that can be analyzed in terms of fractions and shapes (e.g., fold a piece of paper and cut out a triangle, keeping the base on the fold and not cutting that side. Unfold the paper and notice the shape—it is now a parallelogram, or, to say it another way, half a parallelogram is a triangle.)

Cooking is another wonderful context in which to explore fractions and their role in measurement. Cooking activities delight many senses at once and provide a great way to get a "feel" for how much half a cup is, or a quarter of a stick of butter. The relationship of ounces to cups and pounds and of teaspoons to tablespoons (to liquid ounces) are open territory. Having a tasty product at the end is a great reinforcement for the hard work of accuracy and precision in cooking. (See Chapter 5 for some ideas.)

Finally, when posing problems for young children, don't shy away from awkward numbers such as prime numbers or odd numbers that do not divide easily or problems whose answers would take children into the negative numbers. Working with these numbers are incentives for discovering and wrangling with fractions, mixed numbers, and negative numbers. Many of the problems children encounter are too neatly resolved, and not only do not challenge them, but mislead them into thinking that numbers must always be divided evenly with no remainders, or that only positive numbers exist, or that mixed numbers are impossible.

Measurement: Taking the Measure of Things

Measuring things seems to have the same innate appeal and satisfaction to young children as naming things; it's a way of coming to know an object. Measuring with conventional units of measurement such as inches and feet poses the same cognitive challenge as time does; children need to construct the concept of a standard unit and how units relate proportionally to each other (e.g., there are twelve inches to a foot). However, children can and do measure in a qualitative way at first, using any material at hand and then simply counting up how much of the material was used.

Tony's mom spent a busy afternoon constructing a child-sized picnic table. Tony asked to borrow her measuring tape and went around the kitchen measuring everything in sight, proudly pronouncing how long or tall the kitchen table, the chair, and the cabinets were without regard to any unit of measurement. When Tony's mom needed the measuring tape back she suggested that Tony use his blocks to measure the kitchen floor. Tony quickly hauled his crate of blocks out and carefully lined all the middle-sized blocks across the floor. He then counted and recounted the blocks to make sure he had the correct number. He ran down to the basement to announce the number to his mom who then asked him how many of the longer blocks it would take to go across the floor, thus starting him on another project, and one that may well lead Tony to construct a concept of a unit.

Young children also enjoy creating "units" of measurement using their body. Paley (1981) recounts how her kindergartners preferred to measure a classroom rug by counting the number of children that could be laid out on the rug. Children can be invited to grasp the necessity of a conventional unit of measurement by asking them to compare how many of their handspans it takes to "measure" the kitchen table. They can also be asked to try and find relationships between body measurements (e.g., two handspans equal one elbow to finger measure).

Older children may well delight in not only learning about all the different measurement systems we have (e.g., for area, volume), but in learning about the metric system, too.

Estimation and Number Sense

Estimation is a mathematical activity that fits well into the lives of young children, from household activities to car rides to vacations. Estimation activities give children opportunities to develop "number sense," that is, reasonable approximations of answers. Children need plenty of practice in order to develop a feel for "ballpark" estimates, for what is out of bounds, and what variables need to be taken into account in different contexts.

Estimating can begin with the favorite activity of eating. For instance, children can estimate how many rolls will be needed at dinner. How many would be needed if another family came to dinner? Don't forget Joey loves rolls and eats more than his share. Estimating can also be useful in cooking, as in doubling recipes, a process which can be verified by physically doubling amounts (e.g., if the recipe to be doubled calls for 3/4 of a cup, have on hand a two-cup measuring cup as well as a one-cup measuring cup, and simply fill the one-cup to the 3/4 line, deposit the ingredient in the two-cup and then repeat. Figuring out how to "prove" results is a challenging problem for children to solve.).

Data

Children love to collect. In addition to all the physical items with which children inevitably litter their rooms, they can also collect data. Some children find data collection a satisfying means of organizing information. For instance, Nils liked to keep track of how long it took to get ready in the morning, how long it took him to get ready for bed at night, how many times his younger brother fussed at the dinner table, and how long it took him to get his chores done. He kept track of these times and tried to improve his "scores." He even tried to get his younger brother to improve his track record!

Nils' comparisons of times lend themselves to charts and graphs. Inventing his own ways to depict the pertinent information gave him insight into the structure and conventions of graphing.

Another way to invite children to play with their understanding of graphs is to give them a blank bar graph with four bars, two of equal height, one higher and one lower and then ask them to create the story that the graph describes. This exercise is another example of reversibility; instead of asking children to read the graph, they are asked to create the meaning for the graph. This is a good way to see how children understand graphing. Newspapers, especially *USA Today*, are rich in graphs that children can attempt to decipher and even critique in terms of plausibility.

Probability

Jimmy was playing Monopoly with his mom and sister. In anticipation of his turn, he would figure out how many spaces he needed to move to get to a preferred destination. Then he would take the dice in his hands and supplicate them to roll the desired number. He was very disappointed and impatient if he did not roll what he wished and jubilant if he did. He seemed to keep mental track of the results, too, which was evident in his observation that he more often rolled what he wanted if he asked for middle-range numbers than if he asked for numbers that are high (11 or 12) or low (2 or 3). He also noticed that he rarely rolled doubles.

Dice and card games almost seem designed to introduce children to probability. Asking children to predict what they will roll and why certain numbers come up more often than other numbers provide a fertile means for exploring probability issues.

Listening to children's answers clues us as to their underlying theories of probability. For instance, Mary insisted that she would next roll a one because she hadn't yet rolled a one in the game. When gently pushed to explain her reasoning, she said, "Well, it's one's turn. All the numbers take turns and if a number hasn't had a turn in a while then it isn't fair, so then they'll get a turn." For this child, a concept of fairness was the basis for her theory of probability. (Indeed, there are many adults who entertain similar notions as well as many more magical ones concerning special dates and other personally meaningful numbers.)

Asking children to keep track of results when throwing dice can set the stage for a more systematic inquiry into probability. There's also a great game called "Pig" that is sure to invite speculation about probability. Choosing either addition or multiplication, one person rolls two dice and calls out the number, either adding or multiplying the two numbers. The object of the game is to attain the highest possible number before a one is rolled. As long as neither of the dice turns up a one, the players are allowed to add the combined number to their score. Knowing that there is always a chance that a one will appear on the next roll, they can then choose to continue the play or to stop there. If a one does appear, they lose all the "points" they have already accumulated. Children love the risk-taking involved in this game and are unwittingly gaining practice with addition or multiplication (or both, as they must tally up their score to see who won).

Inevitably, the discussion that occurs during this game will begin to focus on the issue of when a one will be rolled. Some Math Trek children engaged in the following discussion about rolling ones: Kenny said, "I just know it's gonna be a one next time, I just know it." "Well Kenny, how do you know that?" " 'Cause there hasn't been a one yet and we're on our seventh round." "So, if you haven't rolled a one six times in a row, it has to be a one the next time? It sounds like you have a rule in mind: 'Ones always turn up if you roll the dice seven times.' Does everyone agree with that rule?" Mike frowned a bit and slowly spoke up: "I don't think so. I don't think we can tell if a one is going to come up next. There isn't any way to tell."

The teacher asked the children to predict what would occur on the seventh roll. The children quickly chose sides, with exclamations of, "It has to be a one next! It has to be!" or "Nah, it could be any number." The teacher ceremoniously rolled the dice. A '2' and a '6' came up. Kenny shook his head in disgust. "Hmmm, what does this mean that a one still didn't come up?" The children shrugged and thought for a while. Leslie piped up, "That you can roll and roll the dice and you never can tell what it will be?" The teacher asked, "So even if I rolled these dice a hundred times, it might not be a one?" Kenny spoke up again: "No, it would have to be a one some time. Maybe on the eighth or ninth roll."

As it turned out, on the ninth roll, a one did turn up, and all those still in the game had to forfeit their points. The children were asked if they had a new rule to determine when a one would come up. Rich said, "You rolled a one the ninth time, so it takes nine turns!" Haley said, "But it could come up any time in those nine rolls." At this point, the discussion ended, but some perplexing issues about probability had been raised and would occupy the children many times in sessions to come.

A good way to invite children to test their hypotheses about probability would be to graph the numbers rolled whenever the game of Pig is played. After several of these graphs were constructed, the issues surrounding the rolling of a one might become clearer. However, the idea that each and every roll of the die is an independent event, with the same probability of rolling each number represented on the die, is a developmental concept that may well take years to fully grasp.

Also relevant to both probability and statistics is the important idea of sampling: How we can get information about a population by taking samples of that population? This subject can be quite intriguing for children to contemplate, especially if the sampling issue is relevant to their lives. The Exploratory Data Workshop at the University of Washington exploited one such relevant case when the m&m and Mars candy company replaced tan candies with blue ones. Children were given covered dixie cups with 12 candies in each. The task was to determine whether the candy came from a new bag or from an old bag. After poking a small hole in the cover, the children shook out one candy and were asked to put a post-it on their cup that announced whether they thought their candy came from a new bag, an old bag, or if the data were inconclusive. The results were graphed. They then put the candy back, shook the cup, spilled two candies, and repeated the decision-making and graphing process. The graphs revealed how many more children were correct when using the larger sample. Thus, children confronted sampling variability and how sampling size affects judgement.

Conclusion

The aim of this part of the book has been to demonstrate the many playful ways in which parents and children can have satisfying experiences with important mathematical ideas. There is one more thing that parents can do for their young talented children. And that is to nurture their own curiosity and interests by continuing to read books, solve intriguing problems, and play around with ideas and materials.[*] In this way, parents can be wonderful role models for their children, modeling playfulness, tenacity, and wonder.

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[*] There are some excellent new books on the market that describe recent discoveries and powerful new ways of thinking about mathematical themes (e.g., Peterson, 1990; Stewart, 1992; and others listed in Appendix A).

First it was fun. I made 9 shapes. Then it started getting boring. Boring means you get tired of doing it. After I did it for a while.

CHAPTER 5: Real-Life Contexts for Mathematical Inquiry

Almost any part of a young child's life can be made to have mathematical relevance. Often, children themselves will detect or invent the mathematical aspect of a situation. Parents can encourage this tendency by being ready to exploit the mathematical potential in their daily lives. We know you don't have much time, but the best questions and problems seem to come from doing activities with your child. Again, paying attention to your child's thinking, questioning, and ways of approaching an activity are your best guides. In this chapter, we give some of our ideas for how to weave math into naturally occurring contexts.

While many specific ideas and activities are delineated on the following table, it should also be pointed out that there are some general mental processes that can be brought to bear on any of the listed contexts. These general processes include organizing, comparing, measuring, finding patterns, and visualizing (e.g., graphing) data.

Some Miscellaneous Ideas on Data Collection

The newspaper is a wonderful source of information. You might also want to invest in some simple instruments for tracking weather, such as an outside thermometer and a rain collector. Your child can pick something of interest to follow, keeping a journal or a graph day by day and posting it on the refrigerator. It might be the batting average or yards-per-carry of a favorite local player, number of homeruns this season for one player or the team, won-lost record over the season, etc.

Your child might like to keep a record of rainfall at your house versus what is published in the paper for the past day, month, and year-to-date. Is the temperature when he/she gets home from school the same as the published prediction?

In your child's name, you could buy a few shares of an inexpensive stock for tracking.

Gardening

Activity	Math Activity	Questions to Ask
Growing from seeds	Pricing the seeds (e.g., 10 for a $1) comparing cost of seeds to cost of seedlings.	How many seeds are in a package? How many in three packages?
	Tracking # of seeds planted versus # of seeds sprouted.	Why don't all seeds sprout?
	Measuring and graphing the growth of plants.	How much faster does a sunflower grow compared to a foxglove? Measure heights every three days.
Mystery planting	Observing and identifying the types of plants.	How can you figure out what kind of plant is growing?
Salad bowl plants	Creating a time-line that shows how each type of plant grows. Figuring out when to plant each type of plant.	Why do some types of plants take longer to grow than others?
Growing plants in rows	Creating multiplication problems.	If there are 7 rows of beans, and 12 plants in each row, how many plants are there all together?
Selling plants or flowers	Figuring out profit and loss by keeping tabs on how much it cost to grow a crop (e.g., tomatoes).	What is the most expensive part of growing tomatoes? How would you figure in labor costs?
Water bills	Analyzing and comparing water bills in summer and winter.	Which month requires the most water? Why? How could water be conserved?
Composting	Keeping track of how long it takes for material to decompose and become fertilizer; compare different strategies for composting (e.g., worm bins versus no worm bins). Compare weekly garbage bags with and without composting.	Which materials decompose most quickly? Why? What is the temperature of the compost pile? How does it compare to the air temperature when you wake up? At noon? Before supper?
Growing sunflowers	Measuring height, predicting growth, growing under different conditions and comparing eventual height, counting seeds and spirals, exploring for patterns.	What conditions would be most favorable for growing sunflowers? Why? Any Fibonacci numbers?
Make a fertilizer mix	Working with proportions.	Look at packages in the store. What do the numbers (e.g., 10-10-5) mean?
Track rainfall	Working with measurement.	How could rainfall be measured? How much water do different types of plants and grasses need?

Food

Activity	Math Activity	Questions to Ask
Making recipes	Learning units of measurement, how to measure accurately, and how to read markers on cups, teaspoons.	How many teaspoons in a tablespoon? How many cups in a quart?
Doubling, tripling and halving recipes	Multiplication of whole and fractional numbers.	How much of each ingredient will we use?
Planning a menu	Dealing with proportions—serving sizes versus number of people served.	What can we do if 12 people are coming for dinner, but the recipe is for 8 people?
	Reading labels on packages Understanding units of measurement.	How big are grams? How many grams in an ounce? In a kilogram?
Nutrition	Understanding relationships between types and amounts of food (e.g., the pyramid).	Why should people eat so many more servings of fruits, vegetables, and grains than fatty foods and sweets?
Big plates and little portions versus little plates and little portions	Experimenting with optical illusions.	Which portion looks bigger, on the big plate or the little plate? How come?
Containers	Experimenting with conservation problems; prediction and estimation (number sense).	Which container holds more? The tall or the squat one? How come? How many nuts do you think are in that cup?
Hamburgers and pizzas	Working with fractions (parts of wholes).	If this hamburger is cut into thirds, and you've already eaten one section, how many are left?
	Working with diameter and perimeter.	Measure a large-size pizza. Is the diameter proportional to the perimeter? How would you find out?
Sharing food	Using division.	If there are 2 dozen cookies, and 8 kids at the party, how many cookies will each kid get? What if mom wants one, too?
Beating egg whites	Understanding volume.	Do egg white peaks get higher in a small bowl or a large bowl? Why do you think that?
Yeast versus other leavening agents	Understanding volume.	Make yeast breads and quick breads and compare height, taste, how long to make, etc.

Food
(continued)

Activity	Math Activity	Questions to Ask
Shopping	Units, quantity and comparison	Which costs less per ounce, the 28 oz. or the 64 oz. size? How much does a helping of your favorite cereal cost? How about with milk? With a banana?
	Keeping a graph of the price of a favorite food, like grapes, that varies over the year.	When is the food most expensive? When is it cheapest? Why?
	Keeping grocery receipts for the month.	How much for meat, fruits, snacks? Which category do you spend the most on?
	Estimating how much money next week's grocery bill will be.	What did you base your estimate on?
	Doing a taste test with tomatoes.	Do they taste better when they're cheap (plentiful) or expensive?
Microwave ovens	Exploring time versus level of power.	How much longer does it take a cup of water to boil if we heat it at 50% versus 100%? What if we start with hot water? What if we heat two cups?
	Estimating time and mass.	How long to heat one hot dog to the way you like it? Two hot dogs?

Birthday Parties

Activity	Math Activity	Questions to Ask
Buying favors	Using multiplication.	If 8 children are invited, and you want to give each child 4 favors, how many favors do you have to buy?
Distributing candy	Using division.	There are 66 pieces of candy in the bag: How many pieces will each child receive?
Cutting cake	Using fractions.	If 14 children are coming to a party, what's the best way to cut the cake? (make a picture first!)
Making paper hats	Making shapes and area.	How can you fold this piece of paper so that you can get the shape you want?
Making chains	Making patterns.	Describe your pattern.
Crafts, e.g., tiles	Working with area.	Can you paint 1/4 of your tile green, but have that 1/4 in different places on the tile?
Ice cream	Using estimation.	How much ice cream do we need to buy to make sure everyone has one scoop? How much extra for "seconds"?
Invitations	Using multiplication and addition.	If we buy invitations and mail them, how much will the invitations cost? The stamps? How much can we save if we make invitations?

Money

Activity	Math Activity	Questions to Ask
Giving allowance	Using multiplication.	If you are getting $2.50 for allowance each week, and you don't spend any for 7 weeks, how much money have you saved?
	Working with equivalence.	Give your child the allowance in different combinations of coins/bills each week: Are all these different ways to give allowance accurate and equal? How come?
Purchasing toys	Using subtraction.	If a toy costs $7.49, and you have a $10 bill, how much change will you get?
Sales	Working with percents.	If this item is 50% off, what will it cost?
Figuring out tax	Working with percents.	At 8% (or 8.2%) tax rate, how much tax will you have to pay on a toy that costs $4.99?
Trading up (working with various denominations)	Working with equivalence.	If you have 83 nickels, how many quarters (or dimes or dollars) would that make?
Saving money in a bank account	Using percents.	If you have $100 in an account that pays 3% interest, how much money will you have in one year? Five years? Ten?

Going to the Zoo

Activity	Math Activity	Questions to Ask
Admissions	Working with addition and multiplication.	If 3 adults and 4 children are going to the zoo, and the admission price is $6.50 for adults, and $4 for children, how much will that cost all together? How about if one of children is under 5 and therefore can get in for free? If an annual family pass costs $50, how many times would your family have to visit to save money with a pass?
Feeding	Using big numbers.	How many peanuts would it take to satisfy an elephant's appetite?
Estimation	Working with measurement.	How would you measure an animal, e.g., an elephant or a lion? Would you include the tail? Height versus width? How tall is that giraffe? Could it eat from a second-story window? A third-story window?
Comparing weight	Experimenting with graphs.	Which animal weighs the most? The least? Which three animals are heaviest?
Observing animals	Investigating perception and pattern.	Are zebras white with black stripes or black with white stripes? Describe the pattern of a giraffe's spots. How many birds can you see in the aviary? Name all the colors of the birds you see in the aviary.

Going Out

Activity	**Math Activity**	**Questions to Ask**
A la carte versus fixed price (e.g., Happy Meals versus component parts)	Making comparisons.	Which is cheapest, to order appetizers and entrees separately or as part of a fixed price?
Gratuities	Calculating percents.	If the bill comes to $10, how much tip should we leave?
Eating at home versus eating out	Making comparisons.	How much does it cost to have a coke at this restaurant? How much does a coke cost when we buy it at the grocery store? How about at the movies? Which place is the most expensive? Why do you think it's most expensive there?
Child's plate versus adult plate	Making comparisons.	Why are prices different? Which is the better "deal?"
Ordering pizza	Making comparisons.	Which is a better deal? Ordering one large pizza or two smaller ones? How come? Can you "prove" it?
Videos versus movies	Making comparisons.	How much money will we save if our family sees this movie on video versus at the theater?
Bowling	Working with division.	If it costs $5 for three games, how much does it cost per game? If it costs $7 for five games, how much per game?
Out to the ballgame	Making comparisons.	How much more would it cost if the four people in the family paid for reserve seats instead of general admission?

Pets

Activity	**Math Activity**	**Questions to Ask**
Surveys	Experimenting with graphs.	What pet is the most popular pet in your family (neighborhood, classroom, etc.)?
	Working with correlation.	Do boys like a certain kind of pet and girls like a different kind?
Pet-watching	Observing pet behavior.	Is your pet nocturnal or diurnal?
Learning about pets	Collecting data.	How do you care for different kinds of pets; how expensive is it to care for different types of pets? Why?
		How long does your pet sleep in a 24 hour period?
Pet buying	Working with comparisons.	How much do various breeds of cats or dogs cost? Why are some breeds more expensive than other breeds? How much at the pound?
Reproduction	Using multiplication.	How often do gerbils have litters? How old are they before they reproduce? How many gerbils do you end up with in a year?

Collections

Activity	Math Activity	Questions to Ask
Collecting coins	Using subtraction.	How old is your oldest coin?
	Comparing coins.	Describe the different patterns you see on the coins.
	Working with comparisons.	What is the worth of your whole collection from face value? Now, look in a catalogue: What is your most expensive coin? How many weeks of your allowance is that?
Collecting stamps	Working with comparisons.	Describe the different patterns you see on stamps.
Collecting rocks	Working with comparisons.	Look at rocks with and without a magnifying glass: What's the difference?

Car Trips

Activity	Math Activity	Questions to Ask
Gasoline usage	Using division and multiplication.	If our car gets 25 mpg on the highway, and gas costs $1.40 per gallon, how much will it cost to drive to Grandma's?
	Using reasoning skills.	If I travel for work, I am reimbursed 26 cents a mile. Why do I get more than the gas costs?
	Collecting and analyzing data.	How many miles per gallon does our car get? Does the rate change when we go on a long trip? How come?
Map reading	Working with addition.	How many miles is it from the place where we live to the place where we are going (add mileage segments given on maps)?
	Working with comparisons.	Can you find a shorter route? Is it shorter on the freeway or the scenic route?
	Determining patterns.	How are freeway numbers determined? Can you find a pattern when you look at freeway numbers on a map of the United States?
Calculating mileage	Working with addition or subtraction.	If freeway exits refer to mileage, and we're at exit 47, how many miles to exit? If we're at exit 54, but are getting off at exit 38, how many more miles to go?
	Working with subtraction.	The trip we're taking is 127 miles. We've gone 51 miles. How many more miles to go? How long do you think that will take?
Speed limits	Working with division.	How long does it take to go 50 miles if the speed limit is 25? 30? 55? 70?
Metric system	Using conversions.	We have 12 miles to go. About how many kilometers is that? (1 km = .625 miles)

CHAPTER 6: Alternatives in Meeting the Educational Needs of Math-Advanced Children—A Smorgasbord

Just as different as are the young children who are math-talented, just so different are the educational options they need. There is no "one size fits all" solution to providing them the support and challenges that will keep them thriving. Parents who are interfacing for the first time with schools are often at a loss to know where to begin.

Whether or not there are special programs for children that are designated as for gifted or highly capable students, schools do offer a number of options that can be shaped to meet the needs of a particular child. Parents need to consider first, a philosophy of education that can guide their efforts and second, some options to look for (or to try to create). This chapter is designed to help parents think in these terms and to develop some negotiating skills so that a real partnership is established with their children's teachers.

Nothing in this book should be taken to imply that there is only one right way to deal with the needs of these children (or any other children, for that matter). There is only one wrong way: To ignore the fact that children differ, to treat them as all alike, to be inflexible, to rob them of joy or confidence.

Not only are there many ways to meet the needs of math-advanced children, but each child needs a mixture of possibilities and experiences—choices creating a full plate selected from the smorgasbord. A rich buffet consists of many dishes from which parents, teachers, and children can select several to match appetites, skills, deficiencies, curiosities, risk-taking proclivities, and so on. Nobody takes just one dish!

Reforms in Teaching Mathematics

Readers will recall from Chapter 1 the fact that recent reforms have revolutionized the ways in which mathematics is taught in children's classrooms. Although no less planful than before, contemporary teaching methods tie children's experience much more closely to their real-life interests than the kinds of instruction most parents remember. Children are encouraged to problem-solve in groups or by themselves and to pursue explorations that enable them to "own" what they themselves have produced. Children's true engagement and sense-making with the number system permits them to invent a repertoire of strategies that reflect their understanding rather than requiring them to learn procedures as if they were following a cookbook or going from one line to another on an income tax return.

New teaching approaches involve materials—everyday objects, blocks and shapes, calculators, computers, and all sorts of measuring devices—and children's talking. Most classrooms hum with conversation as children talk their way through problems, compare strategies with their classmates, ask questions, and from time to time exclaim with pleasure over their discoveries. Teachers spend more time listening and less time talking; more time asking questions and less time lecturing and drilling number facts (although practice has its place); more time helping children to look for patterns and less time asking them to recite memorized facts; more time helping them translate their ideas into words and pictures and less time checking their written computations; and, in general, more time having fun with numbers. Such approaches enable adults to see what children are thinking, whether those adults are teachers or parents. Indeed, the boundaries between "school thoughts" and "home thoughts" become increasingly fuzzy, as real-life problems are the stuff of school, and inquiries begun at school come home.

Parents who have learned how to recognize and facilitate their children's development in mathematical thinking are especially welcome and effective volunteers in classrooms. They also tend to be effective advocates for their children in finding and creating appropriate learning opportunities in the schools. Before considering the school options that parents may want to consider, however, let us introduce some ideas about children and schools that parents may find useful in making sense of the variety of options that exist.

A Guiding Concept: The Principle of the Optimal Match

Every philosophy of education that takes into account individual differences among children works on the notion that, at any time, there is an optimal level of instruction that captures a child's readiness to learn: an optimal match. Only if children are engaged in learning at a level appropriate to their ability and skills, at a level they are almost ready for, is there likely to be a real change in their ways of thinking. Significant learning involves stretching one's mind.

If things go too slowly, boredom and turn-off are almost inevitable; if things go too rapidly or at too high a level, children (especially young children) are likely to become uncertain and avoidant. For math-talented children, the dangers are usually in going too slowly or in too shallow a fashion, with unneeded repetition of what the children already know and too little incentive to become truly engaged with new concepts, to figure out, to experiment, to see connections, to make sense of things. Children who are bored do very little learning. This is especially true of young children, who are almost all inveterate hedonists, ruled by what feels good and what doesn't.

Underchallenged math-talented children who are by nature well behaved may not let their parents and teachers know how turned-off and miserable they are. In contrast, those who misbehave in response to similar feelings may be so irritating that the adults never guess the underlying source of the trouble. (One very bright child of our acquaintance was expelled from first grade for kicking the teacher out of sheer frustration and four years later entered a university from which, a model of deportment, he graduated with a B.S. in mathematics at age 13.)

There are many ways to produce an optimal match in the classroom for math talented children. The essential aim is to pace instructional experiences so that they fit the child's intellectual and personal maturity, thereby producing appropriate challenge, supporting growth, and enhancing the student's energy and motivation. Indeed, one way to tell if an optimal match has been accomplished is to note if the student is in a "flow" state (Csikszentmihalyi, 1990), that is, so engrossed with the activity that time seems to pass by very quickly. Achieving an optimal match requires flexibility, ingenuity, compromise—and effort. The Principle of the Optimal Match, as you've guessed by now, is appropriate for all students, not just those with advanced capabilities.

Fundamental Versus Complementary Components

One useful way to conceptualize components of an optimal match setting for children is to distinguish between those aspects that are part and parcel of the basic instructional program (fundamental components) and those that embroider upon and extend it (complementary components). This chapter basically addresses the former, in essence, the regular school day and those extensions of it that we call "homework." A good many different options that can occur by matching children up with classes that are available will be presented, as well as ways a teacher might adjust what goes on within a given classroom.

There are other possibilities for complementary activities that a school might provide, although most of these tend to be limited to older children. There are, for example, inter-school, regional, and even national academic competitions in mathematics, but few of these involve children in the primary grades. A few younger children will be ready "early," however. The contest for the youngest children of which we are aware is Mathematical Olympiads for Elementary Schools (Dr. George Lencher, 125 Merle Ave., Oceanside, NY 11572). Group activities such as Odyssey of the Mind can begin with kindergarten on a local level; math is embedded in many of the tools the student groups may use to problem-solve in this competition. Such activities need a great many volunteer hours from parents, though they can be very rewarding and fulfilling. Even without such organized activities, an interested and talented parent might start an after-school math club that, for example, seeks out patterns and problems in the real world. There are many such worthy possibilities capable of enriching and extending the experience of talented students, but such complementary activities cannot make up for inappropriate instruction during the regular school day. We owe children joy and challenge and the power of doing something new and difficult—every day.

Acceleration

When children are advanced in any area, their growth has been faster than expected for their calendar age. That's what the term means. To achieve an optimal match, therefore, the challenges presented them must likewise be advanced.

Mathematical knowledge and reasoning, particularly in the primary grades, have basic aspects that are linear and orderly in their sequences. Early-taught skills and strategies are needed in order to proceed to more advanced tasks. Furthermore, even if it weren't absolutely necessary, we do teach topics in relatively predictable order (e.g., we usually teach children fractions before decimals). Within a single class, there are children at many steps along these developmental sequences, with the math-advanced children generally (but not always) leading the way.

For these reasons, it makes good sense to assess each child's developmental status, keep track as the year progresses, and create an optimal match by proceeding along the sequence in what is offered, pacing advancement according to the child's readiness. This process is called acceleration but it does not imply that anyone is pushing a child (as one pushes the accelerator to make a car go faster). Parents can tell people who accuse them of being "pushy parents" that they are not pushing, they are running to keep up with their child!

Even when we offer children what we think of as *enrichment*, deepening and broadening the range of their mathematical concepts and activities beyond the basics, it's important that these activities match their developmental levels. *Advanced enrichment* is probably a better term. (Children should never be given more problems at the same level of the rest of the class simply because they work quickly!)

For many math-talented children, out-of-level assessment measures are needed—tests that are designed for children somewhat older, sometimes considerably older, than those designed for their own grade level. Individual measures administered by school psychologists usually cross numerous grade levels and have plenty of "top" for young children who are advanced. For math-advanced primary-grade children who are being given a group achievement test in school, a good place to start is two grades ahead. Some school assessment can be quite informal—for example, a teacher might administer chapter tests from textbooks borrowed from a teacher in a higher grade. From an assessment, one wants to get a ballpark grade-level estimate of children's math skills and some information about specific skills in both calculation and problem-solving, creating a sense of their "frontiers of development." Some children are basically a chapter ahead; some are a grade ahead; a few are light-years ahead.

Such assessment measures won't tell everything about a child's mathematical reasoning. They won't tell how deeply a child understands a concept. For example, a child who can calculate perfectly well using zeros may not have a clue as to what zeros stand for or how difficult life would be without them. Children who can easily pick out squares and trapezoids may not know their defining features. And a child who can solve a problem using a practiced algorithm may not have enough understanding to get to the same answer by a different route.

And test results certainly won't reveal the specific problems a child is already working on "in her head." Parents and teachers have to listen carefully to children to find out. Sometimes a passing remark will be a clue, when least expected. A notebook and pencil in a pocket ready to jot down such comments can prove to be helpful later when parents sit down to talk with teachers. Indeed, parents sometimes have a great deal to reveal to teachers who, in the hustle of a busy classroom with 30 or more children (some of whom are seriously behind) had no reason to suspect that a thoughtful child was harboring a rich and active mathematical inner life.

Beth had begun reading at age 3 and had understood numbers and categories from a very early age. For example, given a set of plastic blocks at age 18 months, she immediately busied herself sorting them by color, then size, then shape, then color . . . . It was not until January of her kindergarten year that her mother discovered from Beth that the teacher was unaware that Beth could read and multiply. For the next "sharing time," then, Beth read and explained to the class her book, "How Much is a Million?" After that, Beth's teacher provided her more interesting opportunities without her mother's intervention.

Acceleration at the Preschool Level

Few families have access to specialized preschool programs for children whose thinking is advanced. Such programs can offer a wholesome combination of the companionship of other children of about the same chronological and mental ages as well as an educational program that addresses the social skills, fine motor skills, and gross motor skills that young children need.

Most families, however, will want their young children to have some preschool experience, for both the socialization and skill-building opportunities provided. Many working parents, in addition, need day care services, which may be provided by the preschool. Most programs offer a combination of self-directed play and group activities.

The bright child without mental-age companions may be quite lonely in such a school, and the activities provided may not be particularly stimulating. Sometimes it is helpful for the bright child to spend all or part of the day with an older group of children (but, in most preschools, eventually the bright child will be one of the oldest children). Sometimes an aide can take aside some of the more advanced children to read them a story more interesting to them than the one being read to the rest of the group, or to do a more complex activity. Materials that can be used in many ways lend themselves to complex patterns and projects that a child can work on progressively for several days in a row. Because most preschool settings are inherently fairly flexible, and because even the bright child usually has some fine and gross motor skills to learn, preschools are not generally the problem for children that grade school presents. But some children are remarkably unhappy.

Mary was brought to one of the authors for testing because her teachers suspected that she might be autistic. In preschool, every day she took out the same four-piece fruit puzzle and did it over and over, talking to no one. Her parents had also noted a decline in her mood and her play behavior at home. On the Wechsler Preschool and Primary Scale of Intelligence, Revised*, although she was not quite four, Mary was found to be functioning at nearly a seven-year-old level. She wasn't autistic; she was profoundly depressed by the prospect that "school" could hold nothing of interest for her, ever. Once moved to a more stimulating environment, she blossomed into her former sunny self.*

Smorgasbord Options Within the Elementary School Classroom

Once a child's degree of mathematical talent is identified, parents and teachers must still figure out what to do. There are, in fact, a great many ways to accomplish acceleration to achieve an optimal match between the young children's talents and their fundamental instructional program. Here are some strategies.

Compacting the Curriculum

Compacting achieves a productive balance for each child of time spent on the regular curriculum and activities that stretch and extend the curriculum beyond its usual boundaries. For the child who grasps concepts quickly and acquires facts easily, it is appropriate to reduce the number of problems or exposure time to a minimum in favor of other activities that extend concepts and skills to higher levels. A number of authors (see especially Reis, Burns, & Renzulli, 1992; Starko, 1986; Winebrenner, 1992) have presented effective ways of compacting the curriculum as well as making good use of the time thereby saved.

Even before a teacher introduces a concept, some (or many) of the children in the class may already understand quite a bit about the topic. Some teachers let children who want to try it, take the end-of-chapter test before starting each chapter. Children who already know the material to something like an 85% level of mastery can profitably skip it or concentrate on the parts of the chapter they haven't mastered. Unfortunately, many current textbooks present a good deal of content that most children in fact have already mastered, and such pre-testing can avert what would otherwise have been essentially wasted effort. On the other hand, for the child who is exceptionally advanced in mathematical reasoning, even going quickly through grade-level chapters may not be appropriate.

Working Ahead in the Curriculum

If pretesting demonstrates that a child already knows the material, the teacher can continue with chapter testing until a concept or skill pops up that needs work. This is the simplest step to take, but it can be one of the most troublesome if the child progresses (almost inevitably) beyond the textbook for his or her grade. There is a dilemma here: With an optimal level of instruction in one grade, an even worse experience may face the child in subsequent grades. It is essential, therefore, to plan ahead and negotiate with future teachers and school administrators so that the child is not expected to repeat the same work later. That would be the most devastating of all possibilities. Yet, it is untenable to argue that a child should be held back from learning what he or she yearns to master because a bureaucracy is too inflexible to adapt!

Mentoring

For primary-age children, sometimes a math-interested parent, a high-school or college student, or even a student from one of the upper elementary grades can be invited to help with math-related activities for one or more hours a week. Every teacher needs extra hands, and young children usually relish 1:1 contacts, especially with a Big Kid. Our only caution here is to beware of the older gifted children's being imposed upon for these purposes to the detriment of their own opportunities for learning.

Sometimes it is the young math-talented children who are enlisted as mentors for classmates. There are significant ways to make the experience worthwhile for both older and younger mentors. Children can, for example, profit greatly by coaching in teaching techniques such as question asking, wait time, using alternative explanations, and so on. Feedback sessions will give them a chance to reflect on their skills as teachers, help them value and learn from the activity, and make them more appreciative and observant of their own teachers!

Diagnostic Testing Followed by Prescriptive Instruction (DT-PI)

A specific model of accomplishing acceleration for children advanced in mathematical thinking, a more formal version of the last two options described, is outlined for elementary school students by Lupkowski and Assouline (1992, Chapter 4). These authors, translating for young students the approach developed by Dr. Julian C. Stanley in his Study of Mathematically Precocious Youth (Stanley, 1990) describe mentor-based programs working from formal pre- and posttest assessments. The DT-PI is a useful approach to acceleration with math-talented children in elementary grades and provides a careful way to achieve an optimal match with the child's level of readiness, basically by using the curriculum already provided but moving ahead with mentorship support.

Learning Contracts

Learning contracts are written agreements between teacher and child (or a group of children) establishing the parameters of an independent plan, including working conditions and acceptable behavior on the child(ren)'s part. Usually, this will include some parts of the basic curriculum, some advanced enrichment activities that extend the concepts being taught, and some free choice. Children can, in fact, often identify quite accurately what they need to work on, and they can certainly say what they'd like to work on.

Generally speaking, it's a good idea to keep contracts related to the saved-time domain, in this case, math. It may not be fair, for example, to "punish" math-talented children by having them spend extra time working on subjects with which they have trouble. There are, however, creative ways to use mathematics to make connections with tasks the children favor less. One might, for example, have children practice spelling with interesting math words, read biographical material about mathematicians or scientists or about the history of mathematical discoveries, or inventory the manipulatives or science equipment in the classroom (to practice handwriting). Certainly, if the child is eager to work on a major non-mathematics related major project during contract time, there is no reason not to encourage this.

Math-advanced children, just like other children, need the kinds of skills, strategies, and knowledge that take repetition, although they may need less practice than the others. Parents should not be upset if they discover their children spending a few minutes each day in timed practice or written computation of number facts they don't already know automatically. Many bright children are more interested in concepts than skills. Once they understand something they've had explained, they are impatient to move on—but they may not "own" the concept or skill quite yet. The teacher will surely embed some practice with skills and strategies in the assigned problem-solving activities. But math-advanced children shouldn't be permitted to talk their way out of learning number facts and procedures to the point that they become automatic, that is, capable of being retrieved from memory without having to be figured out each time. This automaticity is a great asset as one solves new problems, for it reduces the effortfulness of the process and enables the child to keep focused on the ideas rather than the details. A child who has number facts readily available will be more likely to detect relationships among numbers; for example, the numbers 63 and 81 will immediately be seen as members of the "9 family" by the knowledgeable child. Once these number facts are solidly in place, however, nothing much is to be gained by further practice.

Activities to Extend the Math Curriculum Without Driving the Teacher Crazy

There are a great many sources of ideas for math-related advanced enrichment activities aside from those teachers invent for themselves. One great place to start is the series of publications from the National Council of Teachers of Mathematics, some of which are listed together with other resources in the annotated bibliography at the end of this book. In addition, parents, teachers, and children can devise a whole array of problem-solving activities using materials already at hand or some the child can bring from home (e.g., sports pages from the newspaper, utility bills, road maps, small foreign coins).

A teacher who isn't overstressed by too many other responsibilities may well be able to open up class activities to extensions that challenge the math-talented child. Figure 1 presents a few ideas of assignments that could be used to extend concepts being taught in class. The extensions included here range widely in difficulty level—just as do the abilities of math-talented children in the primary grades. Parents who are willing to work in the classroom a few hours a week may be able to give the attention needed by the children as they encounter such advanced enrichment possibilities, or to work with some of the other children while the teacher attends to the math-advanced child(ren). Parents who are too busy to come to school during the day can help to assemble the sets of materials to be used independently by the children and even come up with further ideas of their own.

Parents are sometimes surprised to discover how limited are the budgetary allotments for materials in the schools. Teachers may be very appreciative if parents furnish resources that they would otherwise have to purchase on their own—for elementary teachers report that they spend surprising amounts of their own money for projects they plan. The annotated bibliography in the appendix might be a good place to start, after conferring with the teacher.

CLASS IS LEARNING	ADVANCED ENRICHMENT ACTIVITY
Combinations to make 10	• Using the 4 operations, find how many ways one can make 10 from combinations using the number 2. • Write equations for these. • Extend the above to make other numbers (5, 12, 41, 64, 0, -2). • Using Cuisinaire materials with the orange rod = 1 rather than 10, what is each of the others worth? On graph paper, graph these combinations. • Combine dice to = 10. For each number, 2-12, find how many combinations one can make with 2 dice to equal that number. What is the probability of getting each combination? Each number?
Adding single digits	• Plan a trip to the state capitol using highway maps. Freeway versus state routes? Other trips? • Use missing addends: If traveled this far, how much is left? • Rate: How many hours by freeway? State routes? Stop for lunch?
Subtracting single digits	• Reframe subtraction as adding negative numbers. Think of as many examples as you can of subtraction and/or negative numbers in real life and make up story problems. • What is the difference in age between the oldest and youngest child in the classroom?
Estimating	• Make estimating jars for class, using manipulatives, objects available. Figure out strategies to increase accuracy (e.g., estimate number of objects by weight, volume, length). How many steps? Bring estimating jars from home. • Ask: What makes estimating easier or harder?
Rounding numbers for place value	• Adapt a board game (e.g., Parcheesi) by requiring that a problem card be answered before each turn. • Individualize pack for each child or group. Try 999 or 9,972 to nearest 10 or 100; 4 x 3 x 2 to nearest 10; -8 to nearest 10. Use fractions, decimals, rounding down.

Figure 1. Ideas for expanding math curriculum after compacting.

(figure continues)

CLASS IS LEARNING	ADVANCED ENRICHMENT ACTIVITY
Exploring 1/2, 1/4	• What is a "quarter?" How much is a quarter of: an hour, a mile, a kilometer, a quart, a cup, a liter, a moon, a year, a dollar, a roll of quarters? • Cut an apple in quarters. Does each weigh exactly the same? • Explore thirds, fifths, and sixths. • Find real-life contexts for fractions (e.g., music, cooking, making change). • Explore multiplying fractions using pattern blocks (e.g., 1/2 of 1/4). • Make patterns with halves and quarters of shapes. • Use geoboards and tangrams to explore fractional parts of shapes (e.g., 1/2 of a rectangle = a triangle).
Dividing by single digits	• Bring in family's utility bills. Average per month? Season? Per person? • Collect small foreign coins if available. Look for value in newspaper and monetary system in the World Almanac: Coin's worth in cents? Dollar's worth in coins? • Play "store" with new prices.
Mosaic patterns	• Use tessellations or quilt patterns. • Analyze tiles in floors or elsewhere in school. • Use computer software to generate tessellations.
Computation worksheets	• Have children figure out why they made the mistakes (bugs) they did or look over class papers (names removed) to find most frequent bugs and report to class (use frequencies and histograms to present data). • Have children write ways to check answers (complementary procedures). • Have children write story problems from these computation problems, or make up more complex problems and write those in story form. • For multi-digit addition problems, provide the answer and leave blanks in the two addends.

Figure 1. Ideas for expanding math curriculum after compacting (continued).

Smorgasbord Options Between Classes

The next several possibilities involve matching children with classrooms to achieve an optimal match.

All-School Math

Using the Joplin Plan (sometimes known as "All-School Math" or "cross-grade grouping"), a school can achieve an optimal instructional match for all students. Everyone in the school does math at the same time; classes are arranged in order of advancement within the curriculum. In schools with more than one classroom per grade, the developmental steps between groups are smaller than full-grade steps, permitting students to move ahead (or behind, if need be) without making great leaps likely to create gaps. The highest group in the school has no limits in instructional level except those appropriate for the children. Because the students in a given group are all at about the same developmental level in math, the teacher and children can target a single topic more effectively than if the children are ability-grouped within the classroom, the teacher spending time with only one small group at a time (Gutierrez & Slavin, 1992).

This approach does, however, require the cooperation of teachers and administrators in agreeing to when and for how long math will be taught. The problem of the older child whose math achievement is at a low level has to be handled tactfully. Finally, the plan doesn't lend itself well to integrated instruction across disciplines, although that can occur at other times during the day. Yet, its advantages are many and it deserves serious consideration.

Cluster Grouping

Classroom assignments can be arranged so that the most highly capable children are placed in one classroom (or possibly two classrooms) at each grade level. In a school with three or four classrooms per grade, typically, the most capable students are distributed among all the classrooms, complicating both their lives and their teachers'. Cluster grouping facilitates the brighter children's working together cooperatively, sparking each others' ideas, and giving their teacher a chance to work with them as a group on more advanced material. Cluster grouping assures that peers who can indeed provide advanced ideas and provocative questions for one another have a chance to do so. Yet, because the children constitute only one cluster within a heterogeneous classroom, they accrue the social advantages of being with a diverse group of classmates. Children need not be formally selected, and a child who is better in some areas than others can be easily accommodated.

Ability Grouping Within the Classroom for Core Instruction, Especially for High Ability Students

Faced with a primary classroom full of children whose developmental levels easily range as much as five grade levels, most teachers already do some ability grouping for reading and math instruction. Even within the highest third of students, the range of achievement can still encompass several grade levels. Compacting, contracts, acceleration, and enrichment in a variety of combinations will still be needed.

Multi-Age Classrooms

Multi-age classroom grouping is a general term for several different approaches to teaching. "Split grade" classrooms generally maintain the distinct curriculum of each grade and can have real advantages for gifted children who are thereby exposed to older children and more advanced curriculum, but only if they are in the lower of the grades involved. To spend two years in a 2-3 split classroom has little long-term advantage, but to spend successive years in the younger grades of a 2-3, 3-4, and 4-5 classroom organization may be beneficial. Eventually, one would expect the child to move up a grade, and this structure provides a gentle path to that outcome. Otherwise, the child ends up essentially repeating the final grade in the sequence.

Other multi-grade approaches present, at least for much of the day, a common curriculum for all the children, although not all are expected to respond at the same level. This kind of approach can potentially provide unlimited "top" for children, but in fact the mixing of ages usually succeeds in so increasing the heterogeneity of the class makeup that providing for individual children taxes teachers severely. Many multi-age classrooms use interest centers that can give children appropriate options, but the most advanced children tend to "use up" such options very quickly unless quite challenging ones are provided. Even among the talented, the less confident children may not choose the challenging options unless asked to do so.

Trading Students: Subject-Matter Acceleration

If a child in a second-grade class would be better instructed in one or more subjects at the third-grade, fourth-grade, or fifth-grade level, this possibility might be considered. One shouldn't limit one's imagination!

We became acquainted with one eight-year-old several summers ago when a professor in the University of Washington's Department of Mathematics called us for advice after discovering the boy in his calculus class. The third-grade teacher of this Vietnamese former boat-child had seen to it that he could complete precalculus courses at a nearby community college the previous year. When he applied for summer school, submitting his transcript, it was assumed that the birthdate must be a typographical error! A combination of subsequent full-time placement in a challenging program for gifted children and tutoring by a high-school calculus teacher to encourage playfulness with practical applications of higher math, permitted this child to complete second-year college instruction in math as well as some science classes by the time he was 11. How many of us would think this was possible, or healthy? It was both, as this friendly, exciting youngster proved as he developed into a strong, warm, gentle, happy, and high-achieving young man.

At the same time, especially in the primary grades, one needs to take into account whether a young child can really keep up with older students in ways other than the central abilities for which the match is needed. One needs to consider whether, for example, the child has the fine motor skills to keep up with the written computation expected and whether the expected level of reading is appropriate. Some allowances may need to be made or extra assistance given. Especially in math, it may be easier to find a good fit in this way than it would, for example, in a writing class, but it's asking for trouble just to plunk a student down in a higher class and expect everything to work out automatically.

Early Entry to Kindergarten or First Grade

Especially for the child whose birthday is within two or three months of the state cut-off, early entry to school might be seriously considered. Most educators are leery of enabling children to cross age barriers, as though calendar age was the single most essential piece of knowledge one could have about a child. Age is, of course, important, but is it all we need to know to place a child in school? Consider that our laws generally dictate that, unless we go through lengthy procedures to override the system, we must admit to kindergarten a child who may have been born three months prematurely on August 31 (due to be born, perhaps, in late November) and must exclude a highly capable child, born full-term on September 1, a child who understands multiplication and is reading at the fourth grade level. (States vary in their cut-off dates.) Is this a reasonable position?

The research evidence overwhelmingly demonstrates the wisdom of welcoming into kindergarten or first grade children who can't jump the birthday barrier but are otherwise mature, intellectually advanced, at least average or better in fine and gross motor skills, and on their way to reading and computation. Careful assessment is necessary and there are many factors to consider—but early entrance is clearly a viable possibility for helping to achieve, for a while at least, an optimal match for children who are eminently ready for school. We urge parents who are considering this possibility to read the more extended discussion in Robinson and Weimer (1991).

Skipping a Grade

While generally less limited by law than in making decisions about early school admission, most educators resist double promotion, or grade-skipping, no matter what the children's academic achievement levels and, indeed, no matter what their social skills or the ages of the friends they spontaneously seek out (Jones & Southern, 1991). Most bright children do seek older friends and share with them academic interests as well as play interests, hobbies, and ideas (Robinson & Noble, 1991).

Advancement by one grade when a child is young may take up a good bit of the slack between grade placement and developmental level for a child who is moderately ahead. Kindergarten, first and second grade are good candidates for skipping when children's academic skills are quite advanced, since in most schools, third grade introduces cursive writing, more complex thinking about math, and transition to more abstract aspects of reading. Later on, eighth grade is often a good choice, since its curriculum is generally not particularly distinctive and it constitutes the transition year before students transfer to high school.

The research about grade acceleration is overwhelmingly positive with respect to capable children whose advancement is in several domains, not just math (Kulik, 1992; Kulik & Kulik, 1984; Rogers, 1992; Rogers & Kimpston, 1992). Children's academic achievement on average profits to the extent of acceleration—those who are accelerated by a grade achieve a whole grade higher than do equally bright children who are not accelerated. And what about their social-emotional health and friendships? The evidence here is again very consistent. There are no overall differences in the self-concept and mental health of children so accelerated, compared with their non-accelerated peers. One might argue, indeed, that since the accelerated children's social comparison groups are composed of children older than they are, they might be expected to see themselves a bit less favorably, a "littler fish in a bigger pond" rather than "bigger fish in a littler pond" (Marsh, 1987; Marsh, Chessor, Craven, & Roche, 1995). If their self-concepts are comparable to those of non-accelerated bright age-peers even under these conditions, they can be seen as doing very well, and if they are shaded a bit by comparison with other capable students, this may be indeed more healthy than seeing themselves as always top of the heap.

Grade skipping won't create an optimal match forever. Children who are a grade ahead as they enter school will be several grades advanced later on. For example, a child whose development is roughly 25% more rapid than that of her agemates may, at age 6, be a grade or so advanced but, at age 12, about three grades ahead. More immediately, primary-grade children who are distinctly advanced in their mathematical capabilities will still need attention, since their skills may already be considerably more than a single grade ahead.

Pull-Out Programs and Resource Rooms

In some school systems, children who are advanced in one or more domains are "pulled out" of their own classrooms for special group instruction, typically from an hour to a day a week. When the curriculum is sufficiently differentiated from and advanced beyond that of the regular classroom to accommodate the needs of the children, such placement can be effective and healthy (Delcourt, Loyd, Cornell, & Goldberg, 1994). In fact, however, such pull-out programs seldom provide core instruction in mathematics and it is still the job of the regular class teacher to make the kinds of adaptations we've talked about.

Furthermore, some pull-out programs, because they must adapt to children from so many different settings, tend to resort to "fun and games," extra field trips, and non-academic activities to keep children's interest high. In doing so, they often do not provide sufficient academic pay-off for bright children to make up for the disruption and expense they cause the system. Furthermore, they also court jealousy from other parents and less advanced students for whom the enrichment activities would be equally appropriate. In an atmosphere like the current one, with tight school budgets and political issues about "elitism," pull-out programs (which are always expensive) are highly vulnerable and often become the target of political pull-and-tug.

Special Classrooms

This is not the place for a discussion of the provision of self-contained classrooms for highly capable children. We point out, however, that in large districts, such programs may well be the easiest way to establish settings in which children's advanced development can be enhanced. In such classrooms, the "normal" pace of instruction is more rapid; the basic curriculum can be covered quickly by all the children; and the kinds of extensions engaged in by only a few children in a regular classroom will be useful for everyone. Children in special schools and separate classes show substantially higher levels of achievement than both their gifted peers not in programs and their gifted peers attending within-class programs, though they may be somewhat more reliant on teacher guidance (Delcourt et al., 1994).

While the range of abilities in self-contained classrooms will still be very high—because the most advanced children may be several grades ahead in one or more domains—strategies that can simultaneously engage children at several levels of competence are likely to be more feasible when the range of ability goes from above-average to sky-high rather than below-average to sky-high! Self-contained classrooms, especially those with class sizes equivalent to those of other classes, are, of course, much less costly than pull-out programs, since they utilize the basic education budget. Furthermore, they are less likely to be political targets because it is clear that the children are working at least as hard as, often harder than, their agemates, on learning tasks that would overtax other children.

Teacher Consultants

Some school districts, both rural and urban, successfully employ specialists to assist regular classroom teachers in planning and executing activities appropriate for gifted children, right in their regular classrooms. Supporting the classroom teacher by bringing books and materials and, above all, new ideas, and sometimes by brief instruction of the children themselves, these specialists can make a significant difference in the ability of schools to meet a child's needs. Two of the nice features of such an approach are its flexibility and the opportunities it furnishes for teacher and specialist to brainstorm and plan. Children need not be formally selected or labeled as "gifted," and children with uneven development can be readily accommodated. Often, there will be one or more unlabeled children who are attracted to the special activity and who thereby reveal themselves as advanced in ways that had not been previously apparent. Staff specialists are add-ons to the school budget and therefore not always available within a district, although such persons often may be found in state or regional service centers if teachers and parents look for them.

Open-Ended Strategies in the Classroom

Thus far, we have touched very little on the real life of the classroom, the interaction of teachers with children, and children with children. It is in the climate, the community, the shared delight in the discoveries of the mind, that a setting is created in which learning can occur. And it is in what Kennedy (1995) has called a "gifted-friendly classroom" that children will want to learn.

In a companion book to this one, *Teachers Nurturing Math-Talented Young Children* (Waxman, Robinson, & Mukhopadhyay, 1996), we describe ways that teachers can establish open-ended activities that are suitable for engaging children at many levels of mastery, including children who are decidedly advanced. Such strategies create child-friendly classrooms in which problem-posing and problem-solving occur in a climate of warmth, acceptance, safety, and respect. Child-friendly classrooms create a community of learners who track and take pride in their own mastery but do not measure their worth by comparing themselves with others. Such classrooms are fun for both teachers and students and constitute good places to grow.

Open-ended strategies rely on the teacher's making materials available and asking thought-provoking questions rather than telling answers. They also provide some time for the children to play, to "mess around" with materials, giving them the chance to pose their own questions. Open-ended strategies are especially effective at helping children become independent in their learning, work well together, and learn to communicate their mathematical discoveries. Such strategies can promote children's views that there are interesting ideas to be discovered, that exploring those ideas is fun, that they themselves are competent to make important discoveries, and that other people's ideas can help. Open-ended strategies invite children to use their abilities in many domains, to respect their own ideas and those of their classmates, to find their own intellectual voices, to have the courage to try new challenges, to make sense of things, and to engage in the whole enterprise in an atmosphere in which mistakes can be taken lightly, as guideposts to what still needs to be figured out. And, above all, in open-ended classrooms, there is playfulness and joy.

Creating Partnerships Between Home and School

If the needs of young children are to be met, firm and positive partnerships need to be forged among the adults who nurture them—their parents, their teachers, their doctors, their soccer coaches, their bus drivers, their babysitters—to create at least reasonably optimal matches in all the domains of their lives. Parents, as we've seen, often have very accurate pictures of their children's development, and sharing these pictures is a good first step. With good will, flexibility, and a good measure of creative spirit, collaborations among the "nurturers" can be successful and satisfying to all, with the result that children are challenged, energized, confident, and healthy.

Too often, however, what should be collaborative partnerships unnecessarily become adversarial relationships, especially between parents and teachers or school heads, beginning in preschool. There are, of course, many reasons why parents and teachers may not see the situation in the same way. Children sometimes behave very differently at home and at school, just as did Mary, the little girl with the four-piece fruit puzzle whose preschool teachers suspected of being autistic. Children who are chatty at home may be shy in a group; children who are bored may misbehave in a variety of inventive ways. In a group setting, the high-energy, actively curious child may be trying indeed, leading the teacher to suspect hyperactivity. The child who seems self-reliant at home may seem isolated and unsociable at school; "sociable" children, on the other hand, may chat the day away without accomplishing very much, unless they are asked to do so. And the nonconforming child who seems "creative" at home may be disruptive at school. Both parents and teachers often need to step back from their own situations to get a better picture of things.

Joe was a "creative" child who loved to experiment. At home, for example, he took liquid paints and made all sorts of combinations, some pretty, some murky. After he tried this at school with the primary colors at the easel, he and his parents were disconcerted at the teacher's objections because, of course, these new mixtures were imposed on the next children to take a turn at the easel. Parents and teacher seemed in full danger of locking horns when Joe suggested that he'd be happy experimenting with markers instead.

Another contrast has to do with the classroom setting itself. At home, people who are beginning to get on each others' nerves can escape to separate rooms, but even a relatively spacious classroom allows for no such respite. The classroom may seem chaotic and overstimulating to a child who prefers order and quiet; other children who are basically bored at home may thrive on the energy level of an active group.

And of course, the spread of abilities and skills in an ordinary preschool or primary school classroom is likely to be very broad, drawing the teacher's attention away from the child who is advanced to the child who is slower. Children who are average or behind in their development tend to be more contented with the regular fare of the classroom. Even dedicated teachers who provide appropriately challenging activities for their more advanced students, only to see those activities rapidly completed and children begging for new ones, may eventually feel truly burdened by eager, curious, talented children.

There also may be philosophical differences between home and school. Educators, having had a great deal of experience with lots of children, may come to believe, for example, that "age is everything," and that it is inevitably socially and emotionally harmful to children to be placed with older students. They may believe that early giftedness is only transient, not a predictor of things to come. And, with the emphasis these days on including slower and even handicapped children in the regular classroom—a practice that indeed is often favorable to those children's development—educators often assume that the two ends of the normal curve are mirror images of each other, and therefore what's good for slower children must be good for the brighter ones as well. What's actually good for slower children who are included in the regular classroom is that they encounter more mature models in their classmates and higher sets of expectations than they would if they were educated separately. Quite the opposite may be true for the brighter children who are not given the opportunity for more advanced instruction or classmate-models who are of maturity levels at least equal to their own.

What Parents Can Do at Home

As partners with the schools, there is much that parents can do at home to help children to adopt habits and expectations that make for excellence in school achievement. They can, first of all, open doors to ideas and encourage "figuring out" thinking as we have described in previous chapters.

Second, and just as important, children deserve to be authoritatively parented—not subjected to a dictatorship (an "authoritarian" home) in which independence of thought and behavior is prohibited, nor to such permissive (or "laissez-faire") parenting that there are no expectations at all. Children need to have firm rules and routines to hang onto, to know that their parents respect and care for them inordinately, at the same time that they expect the children to try their best most of the time. Within limits, effective parents welcome their children's experimentation with independence. Children also need to be able to count on their place in a functioning family. Parents need to be consistent; they need to be benevolently strong; they need to be able to discuss matters without getting into endless battles. Children who come from families with high expectations but simultaneous warm support tend to realize their talents to a degree that other children do not (Csikzentmihalyi, Rathunde, & Whalen, 1993). One sees the roots of such attitudes during even the preschool years.

In this way, children come not only to respect but to trust adults, to assume that rules matter, to hold high standards for themselves, to plan ahead, to seek further goals above and beyond the minimum they can get by with. They know that if they are bored at school or at home, it's up to them to do something constructive about it.

And, above all, children deserve to feel that their parents are partners with their teachers. Parents can certainly acknowledge that situations need to be worked on without criticizing teachers' skills or good will in front of their children. One of our Math Trek second-graders told us, "The school system is not meeting my needs and neither is my teacher." We did not think he had figured this out on his own.

And, finally, although it gets worse later on, even in preschool children begin to learn that it is safest to be just like everyone else. Bright children who feel "different," who have trouble finding classmates who share their interest in math or other domains, often believe that there is something wrong with them and that it is their fault. By striving to hide their talents from others, they may well hide and devalue those talents so that they can't be found again. Parents can help their children to have the courage to accept their own difference by celebrating differences among people in general. When parents explain that children sometimes learn at different rates, when they point out that their own children find some things harder than others (e.g., "Cutting with scissors is hard for you but . . . ."), and when they maintain appropriate expectations and appreciate children's efforts—that is the surest way to tell children that they are "OK."

Negotiating Skills

Parents need to develop a full set of negotiating skills and to use both their own and teachers' insights to create an optimal match between children's readiness to learn and what is offered and expected at school. An excellent little book, well worth reading and rereading, is Fisher and Ury's (1981) *Getting to Yes: Negotiating Agreement Without Giving In*. Most people, as these authors point out, think they must be "hard bargainers" in order not to be taken for softies or pushovers. As a consequence, they tend to force others into taking and defending positions rather than agreeing upon a common goal and moving toward it. These authors suggest a number of steps in negotiating with anyone else that focus on the objectives (in this case, meeting children's needs), help people to understand one another's assumptions, and to problem-solve rather than taking intractable positions. Negotiators can fine tune matters as they go along. Fisher and Ury suggest that the effective negotiator is a "principled negotiator."

Where highly capable children are concerned, it may be very helpful for parents and teachers to confer early in the year, to establish goals and to brainstorm about possibilities that may be both feasible and helpful. A non-exhaustive list of options has been described earlier in this chapter, but some are more feasible than others in particular schools and preschools, and all are a better fit for some children than others. Before they get together, parents and teachers can usefully sit down quietly by themselves with pencil and paper and perhaps a cup of tea or coffee, and list as many specific possibilities as creative imagination allows—within reason, of course. The list can include some alternatives that will require the parent's assistance—such as teaching some missing skills a child needs to take the next big step, going beyond school resources if a project demands it, looking over a child's projects, reading the same advanced book the child is assigned so that it can be discussed, etc. Parents can also help children avert boredom by assuring that they always have with them an interesting book to look at when they've completed expected activities. (As the children proceed through elementary school, these should be non-fiction books, so they will look like "school work.")

At the conference, parents' and teachers' lists may spark still more ideas, but in the process, some are likely to emerge as more feasible and helpful than others. If either parents or teachers have an idea they'd like to try, especially if it is a significant change from usual practice (such as moving a four-year-old in with a kindergarten group), it is often wise to ask for a trial for, say, six weeks or to a specific holiday break, followed by a telephone or in-person conference. Such experimentation won't back anyone into a corner or ask a long-term commitment of someone who may see the move as risky or "against regulations." Indeed, if it doesn't work, everyone will have gained good information about the experiment and have further ideas about what to try next.

And, if glitches do occur in the ordinary course of things, or in the course of one of the experiments, a conversation is again in order. It's important to find things to praise about what the teacher has done and to thank him or her for the effort the experiment has taken. Then, one can describe, non-critically, how things have seemed from the home perspective, ask for the teacher's perceptions, and most especially, for feedback and advice as to how to help. Is there another way to accomplish this goal without upsetting the applecart? Maybe another experiment is in order?

Compromises

One of the realities of life is that there is no perfect world for the child who does not fit the norm. Some solutions have more advantages than others, but it is impossible to have it all. Each family and each school, and each pattern of children's abilities and needs, dictates some priorities, and no one solution is best for everyone. Parents and teachers try, through their partnerships, to reach the best compromise.

Concepts of fairness are critical here. Especially in public schools, it is important that parents not ask for a better education for their children who are advanced than other students are given—not more field trips, more science labs or computers, smaller classes, or more experienced teachers. What children need and deserve is the most appropriate selection from what reasonably can be made available. We all need to remind ourselves that being fair to children does not mean that we do the same for everyone—it means that we try equally hard to meet everyone's needs to the degree that we can. Parents already know this. They don't buy each of their children new shoes just because one child has outgrown his, and certainly they don't buy the same size for everyone. They buy shoes that fit.

Conclusion

Achieving an optimal match for math-advanced children can and should take many forms. This smorgasbord of strategies merely scratches the surface. As mathematics education is being reformed (e.g., National Council of Teachers of Mathematics, 1989), its greater flexibility, its multiple goals and strategies, and its real-world orientation are all beautifully suited to a learning community in which math-talented children can thrive.

Parents who watch and listen to their children—who engage them as "mathematicians"—will be full of ideas to try at home and at school. This chapter has described some ways that schools can organize to meet children's needs. But what happens in the home remains of primary importance—parents and children sharing the excitement of exploration of ideas about numbers. Math-talented children are particularly fun to listen to. They are very much worth the trouble.

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Winebrenner, S. (1992). *Teaching gifted kids in the regular classroom*. Minneapolis, MN: Free Spirit.

Appendix A

Annotated Bibliography

Annotated Bibliography

Children's Literature

Anno, M. (1983). *The mysterious multiplying jar* (1-6). New York: Philomel Books.

Wonderful pictorial illustration of factorials.

Anno, M. (1982). *Anno's math games I*. New York: Philomel Books.
Anno, M. (1985). *Anno's math games II*. New York: Philomel Books.
Anno, M. (1987). *Anno's math games III*. New York: Philomel Books.

These books include a variety of intellectual games that invite children to see how mathematics permeates the world.

Anno, M. (1989). *Anno's hat tricks*. New York: Philomel Books.
Anno, M. (1990). *Socrates and the three little pigs*. New York: Philomel Books.

These two books explore probability.

Burns, M. (1994). *The greedy triangle* (K-3). New York: Scholastic.

The story of a triangle that magically changes shape in order to see what it's like to have more sides and angles. Good for discussion of how shapes are part of our world.

Draze, D. (1990). *Can you count in Greek: Exploring ancient number systems* (1-6). San Louis Obispo, CA: Dandy Lion Publications.

Explores Egyptian, Babylonian, Roman, Greek, and Mayan number systems.

Eichelberger, B., & Larson, C. (1993). *Constructions for children: Projects in design technology*. Palo Alto, CA: Dale Seymour Publications.

Instructions for projects from bridges to clocks with gears.

Schwartz, D., & Kellogg, S. (1985). *How much is a million*. New York: Scholastic.

Playing with astronomical numbers and what they look like.

Schwartz, D., & Kellogg, S. (1989). *If you made a million*. New York: Scholastic.

More playing with astronomical numbers.

Scieszka, J., & Smith, L. (1995). *The math curse*. New York: Viking Press.
Tompert, A., & Parker, R. A. (1990). *Grandfather Tang's story*. New York: Crown Publishing.

Story of animals transforming; based on tangrams.

Books on Children's Mathematical or Scientific Learning

Duckworth, E. (1996). *The having of wonderful ideas and other essays on teaching and learning* (2nd ed.). New York: Teachers College Press.

Duckworth has wonderful ideas about how children learn and how to understand children's learning and sense-making.

Gallas, K. (1995). *Talking their way into science: Hearing children's questions and theories, responding with curricula*. New York: Teachers College Press.

Reflections and examples by a first/second grade teacher.

Ginsburg, H. (1989). *Children's arithmetic: How they learn it and how you teach it*. Austin, TX: Pro-Ed.

This book explains how children's mathematical thinking develops, beginning in infancy. Great examples of how children invent meaningful ways of approaching numbers and calculations.

Kamii, C. (1984). *Young children reinvent arithmetic*. New York: Teachers College Press.

Kamii, C. (1989). *Young children continue to reinvent arithmetic: Second grade*. New York: Teachers College Press.

Kamii, C. (1993). *Young children continue to reinvent arithmetic: Third grade*. New York: Teachers College Press.

These three books describe in detail how children invent their own mathematically meaningful ways to compute with numbers. Based on the author's own classroom research.

Papert, S. (1993). *The children's machine*. New York: Basic Books.

Papert, one of the inventors of Logo, a computing program for children, writes here about his vision for how children learn best.

Schifter, D., & Fosnot, C. (1993). *Reconstructing mathematics education*. New York: Teachers College Press.

How several teachers changed to a more open-ended and constructivist approach towards teaching math.

Curriculum Materials

Burns, M. (1987). *Math solutions K - grade 3*. New Rochelle, NY: Math Solutions Publications.

Detailed descriptions of lessons that the author conducted in classrooms over several sessions each.

(Note: Marilyn Burns is a prolific author, with many worthwhile titles.)

Childcraft. (1988). *The how and why library* (Vol. 13: Mathemagic). Chicago: World Book.

This book explores the history and magic of mathematics through stories, puzzles, and activities.

Garland, T. H. (1987). *Fascinating Fibonacci: Mystery and magic in numbers.* Palo Alto, CA: Dale Seymour.

Explorations of the Fibonacci sequence in nature, anatomy, art, and architecture.

Haag, V., Kaufman, B., Martin, E., & Rising, G. (1995). *Challenge: A program for the mathematically talented* (Grades 3-6). Reading, MA: Addison-Wesley.

This is a program for teaching logic through puzzles and problems as well as other aspects of mathematical thinking and problem-solving.

Kaye, P. (1987). *Games for math.* New York: Pantheon Books.

An inventive educator has come up with great ideas for how to play with math.

Magarian-Gold, J., & Mogenson, S. (1990). *Exploring with color tiles* (K-3). White Plains, NY: Cuisenaire.

Activities using tiles to explore operations, perimeter, and probability.

National Council of Teachers of Mathematics (1990-1996). *Curriculum and evaluation standards addenda series*. Reston, VA: Author.

This is a wonderful series that deals with number sense, patterns, making sense of data, spatial sense, etc.

Ritchhart, R. (1995). *Making numbers make sense: A sourcebook for developing numeracy*. Reading, MA: Addison-Wesley.

This resource book uses interesting problem-solving contexts in which to investigate traditional topics in math such as place value, measurement, and statistics.

Stenmark, J., Thompson, V., & Cossey, R. (1986). *Family math* (K-8). Berkeley, CA: Lawrence Hall of Science.

Activities for all areas of mathematics that can be done at home with parents and siblings.

Winebrenner, S. (1992). *Teaching gifted kids in the regular classroom*. St. Paul, MN: Free Spirit Publishing.

Strategies and techniques to meet the academic needs of gifted children in regular classrooms.

NOTE: Dale Seymour and Cuisenaire put out catalogues that include many excellent titles. TERC, a research group, also publishes books that detail curriculum units to be used in conjunction with hands-on math and science learning. TERC's address is: 2067 Massachusetts Avenue, Cambridge, MA 02140

Books to Satisfy Your Own Mathematical Curiosity

Peterson, I. (1990). *Islands of truth: A mathematical mystery cruise*. New York: W. H. Freeman.

Peterson, I. (1988). *The mathematical tourist: Snapshots of modern mathematics*. New York: W. H. Freeman.

Stewart, I. (1992). *Another fine math you've got me into*. New York: W. H. Freeman.

Appendix B

Math Games

Math Games

LOTS OF BOXES

Object of the game: This is a game for two people. The idea is to make a bigger rectangle than your partner.

Directions:

1. Each partner takes a piece of grid paper and a pencil.

2. One person takes the die and throws it. The number on the die tells you how *long* your rectangle will be. Now draw it. Then you throw the die again, and that will tell you how *high* your rectangle will be. Now draw it, and finish your rectangle. How many little boxes are in your rectangle? That's your score. Write it down.

3. Now it's your partner's turn to do the same.

4. Whoever has the bigger score, wins.

Play as many times as you like!

Questions to ask:

1. How did you figure out how many little boxes were in your rectangle?
2. Could you find an easier way to figure it out?
3. Could you write a number sentence to show how many little boxes there are altogether?
4. What's the smallest rectangle you could make by throwing the die twice?
5. What's the biggest rectangle you could make?

Challenges:

1. Use two dice each time you throw!
2. Devise two ways to figure out how many squares there are in your box.
3. Write number sentences or equations for each way.
4. Write a formula for finding out how many squares there are no matter what you roll.

WRITE A STORY FOR . . .

1. Write a story for this number sentence:

   0 - 3 = -3

2. Now draw a picture to go with your story!

WHAT IS YOUR NAME WORTH?

If A = one penny, B = two pennies, etc., what is your first name worth?

What is your last name worth?

How did you figure it out?

What is the most expensive name you can think of?

Think of a name that costs exactly forty-three cents!

Can you think of any variations on this?

SWITCHING PLACES BOAT PROBLEM

First, draw three circles big enough to fit color tiles on to:

O O O

Take 1 blue and 1 yellow tile and place them on the two outside circles. Imagine that the paper is a boat with three places, and the blue person wants to switch places with the yellow person. They can only switch by moving one position at a time, and in one direction. They can also jump over one person, but only one of an opposite color.

After you have figured this out, draw 5 circles and use two blue and two yellow tiles, leaving the middle circle blank.

Record how many moves it takes to switch sides.

Try this game with 7, 9, and 11 circles.

Record the following information in a table: Places in the boat. How many moves it took to switch places. Do you notice any patterns?

Chip-Trading Game*

Materials: Poker chips in four different colors (e.g., yellow, blue, green, and red). Playing sheets divided into four columns with headings in the same colors as the chips, from right to left: Yellow, blue, green, and red. One or two dice (depending on the base used). 2-5 people may play at a time.

1. Choose a "Land" to play in, e.g., the "Land of Threes" (In this game, "land" is a metaphor for the base in which the game is played).

2. The object of the game is to acquire a red chip. In the Land of Threes, wherever three or more of any color are accumulated, they must be traded in for the next color, as that next color represents the next place value. For example, if the number 4 was rolled, the player would take four yellows and would then need to trade three of those yellows in for a blue, so that the player would have one yellow and one blue on the color-coded playing board.

3. Each player takes turns throwing the die until someone acquires a red chip.

4. This game may also be played with a "banker." The banker presides over trades by asking the player what she wants traded and why. In this way, the banker elicits the other player's reasoning.

5. The game may also be played in reverse: Each player starts with a red and subtracts the number thrown on the die for each play, until there are no chips left on the playing board.

* Davidson, P., Galton, G. K., & Fair, A. W. (1975). *Chip-trading activity*. Fort Collins, CO: Scott Resources.

(Featured in school catalogues such as Dale Seymour and Cuisenaire.)

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